Session 5

Indirect Measurement and Trigonometry

Key Terms in This Session

Previously Introduced

- ratio
- scale factor

New in This Session

- proportion
- similar triangles
- tangent

Introduction

How do people determine lengths when they can't use standard measuring tools, such as a tape measure? How do they find the distance across a river if, for example, they don't have a boat? Measurements of very large objects or of long distances are often made indirectly, using similar triangles and proportions. Such indirect methods link measurement with geometry and number.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will do the following:

- Explore a number of methods for indirect measuring, such as using similar triangles, shadows, and transits
- Learn about right-triangle trigonometry
- Learn about the relationships between steepness, angle of elevation, and height-to-distance ratio (tangent ratio)
- Use trigonometry ratios to solve problems involving right triangles

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Note 1. Materials Needed:

- Drinking straw*
- Metric ruler*
- Protractor*
- Pushpin*
- Tape
- Ruler
- Tape measure
- Scientific calculator or trigonometric function table
- Trundle wheel (a plastic wheel, usually graduated in 5 cm intervals, designed to measure lengths by counting the number of clicks, each of which equals 1 m). It can be purchased from:

ETA/Cuisenaire, 500 Greenview Court, Vernon Hills, IL 60061; Phone: 800-445-5985/800-816-5050 (Customer service); Fax: 800-875-9643/847-816-5066; http://www.etacuisenaire.com

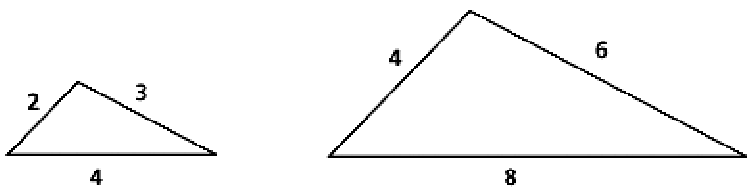
* You'll need one of these for each homemade transit you make in Part A.

Part A: Indirect Measurement With a Transit (40 min.)

Similar Triangles

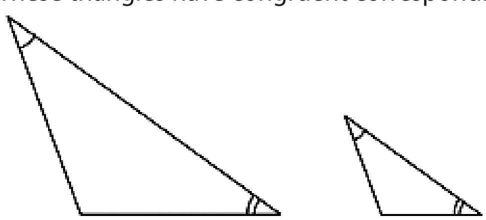
One way to measure indirectly is to use similar triangles. In similar figures, corresponding angles are congruent, and corresponding sides (or segments) are in proportion. In similar triangles, however, one or the other will suffice. In other words, if the corresponding sides alone are in proportion, the triangles must be similar.

These triangles have proportional corresponding sides (a ratio of 1:2):



Likewise, if the corresponding angles alone are congruent, the triangles must be similar. Notice that because the sum of angles in a triangle must be 180 degrees, we only need to know that two of the corresponding angles are congruent to know that they are similar. **[See Note 2]**

These triangles have congruent corresponding angles:



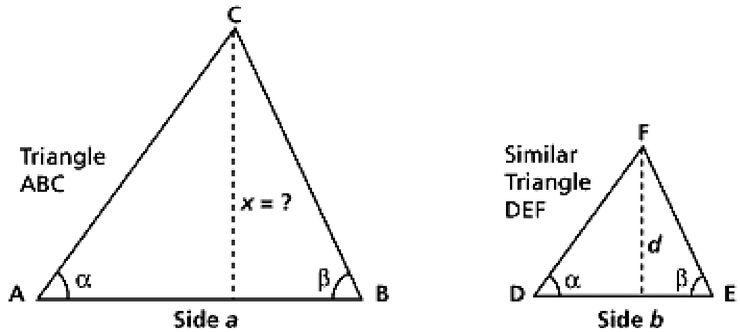
Using Similar Triangles

If we have actual measures for at least one of the similar triangles, and we also know the scale factor or ratio that links the two triangles, we can use proportional reasoning to determine the measure of the unknown side(s) on the other triangle. This unknown side corresponds to the object or feature of the object we're trying to measure.

Note 2. To further explore the concept of similarity, go to www.learner.org/learningmath and find *Geometry*, Session 8.

Part A, cont'd.

In the following example, triangle ABC is similar to triangle DEF. To find the length of *x*, you set up a proportion as shown below. (Notice that *x* is perpendicular to side *a*, which will help us with calculations.)

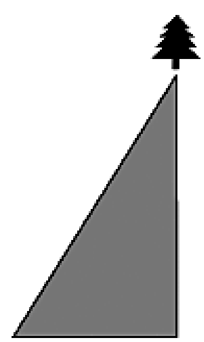


$$\frac{x}{d} = \frac{\text{Side } a}{\text{Side } b} \quad \text{or} \quad x = \frac{\text{Side } a}{\text{Side } b} \cdot d$$

When we use similar triangles to measure indirectly, we usually collect some measurements from a triangle that can be imagined using landmarks (the lengths of some of the sides and/or the measure of some of the angles) and then draw a similar triangle on paper. We need adequate information to make sure that we are dealing with a unique triangle; knowing the length of one side of a triangle would not be enough information to draw a similar triangle, but knowing two angles and one side would be. We also need to know the ratio that links the corresponding sides in the two similar triangles as you've seen above. We will explore this process in greater detail in the next section.

Measuring Distances

When measuring outdoors, it's relatively easy to measure the size of different angles. For example, if we want to make a scale drawing of a particular location, we can work with the angles formed by imaginary lines joining trees, buildings, and other landmarks. To take such horizontal and vertical angle measurements, civil engineers use an instrument called a transit. We will use a homemade transit to measure horizontal angles.



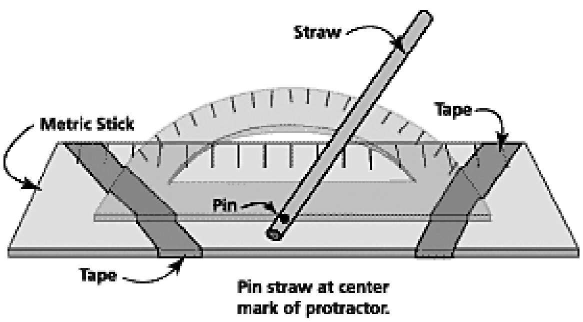
Suppose we want to find the distance across a field to a tree. We'll make the base of the imaginary right triangle the side of the field where we're standing, and the tree the opposite vertex of the triangle. We can imagine an infinite number of triangles. At right is one possibility: **[See Note 3]**

Even when you measure indirectly, you still have to take some direct measurements. First you must establish the measure of at least two of the angles in a triangle. (Why don't you have to measure the third angle?) Then you must physically measure one side of the triangle so that you can establish a proportional relationship between the side you've measured and the corresponding side in the similar triangle.

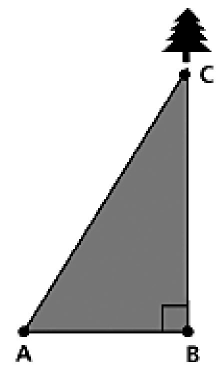
Note 3. It may not be clear that an infinite number of similar triangles can be drawn to correspond to the original triangle. This is because a ratio can be established between corresponding sides, no matter what the size of each of the triangles (as long as the angles in both triangles are congruent). If you are working in a group, discuss this idea and make sure that everyone in the group understands how to set up a proportion in which one ratio is equal to another ratio.

Part A, cont'd.

To take such measurements, you can use a homemade transit for the angles and a trundle wheel for the distance between them. You can make a transit with a straw, a metric ruler, a protractor, a pushpin, and some tape.

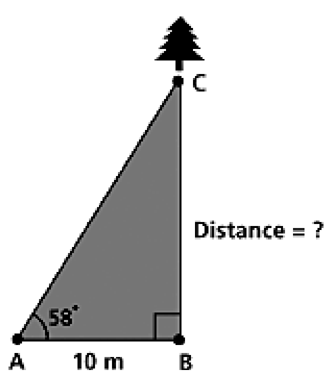


To use the transit, stand at each endpoint of the base of your imaginary triangle and hold the transit at eye level. Move the straw to line up with the object under scrutiny, and read the angle measure off the protractor.



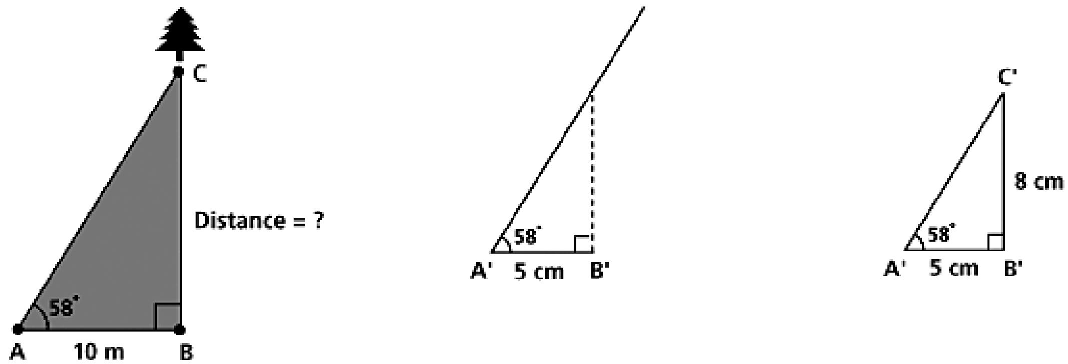
To try this yourself, go outside and find a tree across a field or parking lot. Set up an imaginary line along the side of the open space opposite the tree by putting markers down for points A and B. From point B, the tree should appear to be directly in front of you. Using the transit, sight the tree (which will be point C) across the field. You want point B to be perpendicular to both points A and C (namely, $\angle B$ should be a 90-degree angle). You may have to move point B a bit on your base line to make sure that you have a right angle. Next, use the trundle wheel to find the distance between points A and B. Make sure that the distance is at least 10 m (you may have to move point A). Finally, stand at point A and sight the tree (point C) in the distance, using the transit. Draw a sketch of $\triangle ABC$ and record the angle measures for $\angle A$ and $\angle B$ and the actual distance between points A and B. Notice that we don't know the distance between points B and C and between points A and C at this time.

Now your triangle might look something like this:



Part A, cont'd.

Next, you need to draw a triangle similar to ΔABC (we'll call it ΔA'B'C'). The length of AB determines the similarity ratio or scale factor, so you want to pick a convenient scale; for example, 1 cm on the drawing could equal 2 m in the real world. Use your scale factor to calculate the length of A'B' and to draw ΔA'B'C'. Next, measure B'C' on your scale drawing and set up a proportion to find the corresponding measurement (BC) in the original triangle. Here is an example using the scale 1 cm:2 m:



$$\frac{AB\ (10m)}{A'B'\ (5cm)} = \frac{BC(x)}{B'C'\ (8cm)}$$

The length of BC is 16 m.

Problem A1. Use the technique of similar triangles to determine the distance between two objects outside.

a. Use your transit to help you construct a right triangle and measure angles. Remember to measure the distance between your two sight points. Sketch a triangle that represents the angle and side measures.

b. Decide on a scale factor (similarity ratio) and draw a similar triangle. (You will need to use a protractor.)

c. Find the approximate distance to your object from one of your sight points.

Video Segment (approximate time: 5:53-8:04): You can find this segment on the session video approximately 5 minutes and 53 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Susan and Jonathan indirectly measure the distance across a parking lot to a nearby tree. They use a trundle wheel and a transit to measure one side and one angle of a triangle, and then set up a similar triangle to calculate the unknown distance.

What are some advantages and disadvantages of this type of indirect measuring?

Problem A2. Why do you think we use similar triangles rather than other similar figures for this indirect measurement? **[See Tip A2, page 109]**

Problem A3. In Problem A1, what other distances could you find indirectly using your similar triangle?

Problem A4. Explain in your own words the relationship between similarity and indirect measurement.

Part B: Measuring Heights of Tall Objects (35 min.)

Measuring With Shadows

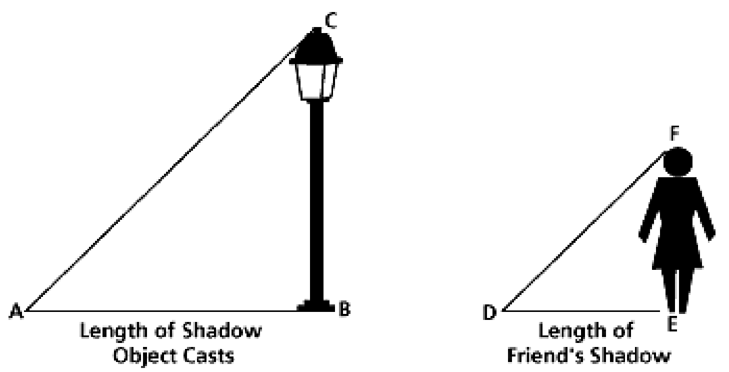
What other methods exist for measuring indirectly? One such method is to use shadows. For tall objects that are difficult to measure directly, such as skyscrapers and giant redwood trees, shadows can be very useful, as they lie on the ground and are fairly easy to measure. The "shadow" method also relies on similar triangles.

Problem B1. Go outside on a sunny day with a friend, and find a tall lamppost that is casting a shadow. (If you can't find a lamppost, use another tall object.) You need to be able to measure the length of the shadow, so be wary of shadows that fall onto roadways. A large parking lot or field is ideal. **[See Note 4]**

a. Measure the length of the shadow of the lamppost. Make sure that the lamppost is on flat, level ground, since the lamppost and the shadow should be perpendicular to each other. Record the measurement.

b. Now measure the shadow your friend casts on level ground.

c. Make a sketch of the lamppost and shadow and label what you know. Also draw a right triangle that shows your friend and his or her shadow as the legs of another triangle. Again, label what you know. Your sketch should look something like this:



Problem B2. Why are the two triangles (lamppost/shadow and friend/shadow) similar? Think about the angles in each of the triangles. How were those angles formed? **[See Tip B2, page 109]**

Problem B3. If you know that the triangles are similar, how can you find the height of the lamppost?

Note 4. This activity should be conducted on level ground. It is important that the object of interest (e.g., a tree or person) and its shadow are perpendicular to each other so that the angle formed in both cases is 90 degrees.

It is also important that you measure the lengths of the shadows at the same time of day. The rays of the Sun hit the tops of these objects at a specific angle, depending on the time of day, and in order for the triangles to be similar, the angles must be congruent.

Note that this method only works with sunlight, and won't work with lamplight. Rays of sunlight are, for our purposes, parallel. Lamplight rays come from a point source, so they radiate. The same object will cast a different length shadow depending on its distance from the point source.

Part B, cont'd.

Problem B4. Determine the height of the lamppost. Discuss the different proportions that you might use to calculate its height. **[See Note 5]**

Take It Further

Problem B5.

a. Why is the height of the lamppost a derived measure (i.e., measured indirectly)?

b. Let's assume that your measurements were accurate to the nearest 0.25 m. Use proportions to calculate an upper and lower limit on the height of the lamppost.

c. What do you think is the best value for the height of the lamppost?

Problem B6. Imagine that you have a tall tree in your yard that needs to be cut down. You want to make sure that the tree won't hit your house when it falls. How might you approximate the height of the tree?

Video Segment (approximate time: 8:28-11:00): You can find this segment on the session video approximately 8 minutes and 28 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this segment, Katy and Lombi use the method described in this section to find the height of a tree in the schoolyard. They set up a similar triangle by measuring the length of the shadow of a meterstick.

What are some of the advantages and disadvantages of using a shadow to measure lengths?

Note 5. If you are working in a group, discuss the similarities and differences between measuring shadows and using a transit to find an indirect measure. You might want to consider two important facts:

- Similar triangles must have congruent angles. (How do we guarantee this in both cases?)
- The corresponding sides must be in proportion. (How does this occur in both cases?)

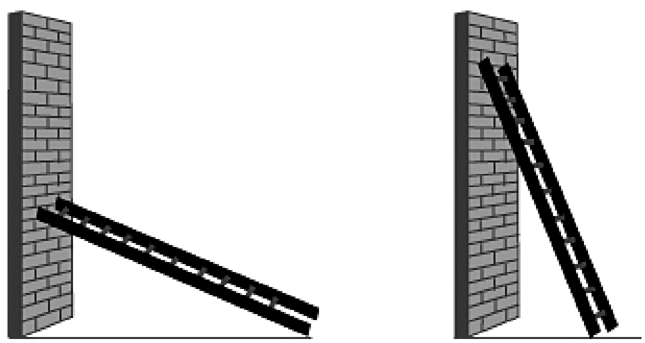
Part C: Steepness and Trigonometry (45 min.)

Measuring Steepness

Since early times, surveyors, navigators, and astronomers have employed triangles to measure distances that could not be measured directly. Trigonometry grew out of early astronomical observations, such as those of Hipparchus of Alexandria (140 B.C.E.). The word trigonometry comes from the ancient Greeks and literally means "triangle measurement." In Part C, we explore right-triangle trigonometry, which provides us with another method of deriving angles and lengths when we can't measure directly. Instead of using similar triangles, trigonometry is based on ratios of the sides of a right triangle that correspond to various angle measures. **[See Note 6]**

While in this session we are only using the tangent ratio to measure indirectly, there are a total of six ratios associated with any angle in a right triangle: sine, cosine, tangent, cotangent, secant, and cosecant. Earlier mathematicians recorded these ratios and corresponding angles in tables they used for calculations, but today most of us use a scientific calculator to find this information.

The following drawings show two side views of the same ladder leaning against a wall:



Problem C1. Describe the differences between the two ways the ladder is positioned against the wall in the above drawings:

a. What problems might occur if the ladder is very steep?

b. What problems might occur if the ladder is not steep enough?

As the steepness of the ladder changes, the following measures also change:

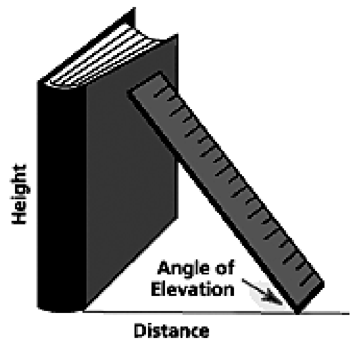
- The height on the wall that is reached by the top of the ladder
- The distance between the foot of the ladder and the wall
- The angle between the ladder and the ground (often called the angle of elevation) **[See Note 7]**

Note 6. When considering an angle and the height-to-distance ratio formed by that angle, it is common to name the angle with a Greek letter. The first letter in the Greek alphabet is α (alpha), the second letter is β (beta), and the third letter is γ (gamma). In this session, we frequently refer to $\angle\alpha$.

Note 7. Take your time working through the problems in order to make sense of the relationships, as future questions will build on what you learn here.

Part C, cont'd.

Problem C2. Let's investigate different levels of steepness by using a ruler to represent a ladder, and an upright book or box to represent a wall, like this:



The angle between the height *h* and distance *d* must be 90 degrees. As the angle of elevation increases, what happens to the height and distance?

For a non-interactive version of this activity, use a ruler to represent a ladder and a book to represent an upright wall. Fill in the chart with five different sets of measurements.

Try It Online! **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Measurement* Web site at www.learner.org/learningmath and find Session 5, Part C.

Ratio (height:distance)	Angle of Elevation

Problem C3. What patterns do you notice between the height-to-distance ratios and the angles? **[See Tip C3, page 109]**

As you've now seen, there are several ways to measure the steepness of a ladder. You can measure the angle $\angle\alpha$ or you can find the height-to-distance ratio, which is entirely dependent on $\angle\alpha$ and not the length of the ladder. This ratio can be expressed as a fraction, decimal, or percent. The ratio *h:d* is also called the tangent of $\angle\alpha$, or $\tan \alpha = h/d$. It is a derived measurement rather than a direct measurement.

Problem C4. What happens to $\angle\alpha$ as the height-to-distance ratio increases?

Problem C5. Why must the angle between the height and the distance be 90 degrees?

Part C, cont'd.

Problem C6. Use the ratios below to determine the tangent of the various angles. Record the relationships in the table using tangent notation. For example, if you have an entry with a height-to-distance ratio of 3/3 and an angle measure of 45 degrees, you can record this relationship as tan 45° = 1.

Ratio (height:distance)	Angle of Elevation	Tangent Notation
3.8/10.3	20°	
5.5/9.5	30°	
7.8/7.8	45°	
9.5/5.5	60°	
10.6/2.8	75°	

Examining Ratios and Angles

Problem C7. Use a protractor or angle ruler and a ruler to make side-view scale drawings of a ladder leaning against a wall for each of the following situations. Label the measures of $\angle\alpha$ and the lengths *h* and *d*, and find the height-to-distance ratios. Record your answers in the table below, or use the Ladders Worksheet on page 108.

a. $\alpha = 45°$

b. $h = 2$, $d = 1$

c. $\alpha = 30°$

d. $h = 1$, $d = 2$

e. $\alpha = 60°$

Problem	Measure $\angle\alpha$	***h:d*** **Ratio**	Ratio as Decimal
a	45°		
b		2:1	
c	30°		
d		1:2	
e	60°		

[See Tip C7, page 109]

Problem C8. Examine Problem C7 (b) and (d) above. What do you notice about the *h:d* ratio and the measure of $\angle\alpha$?

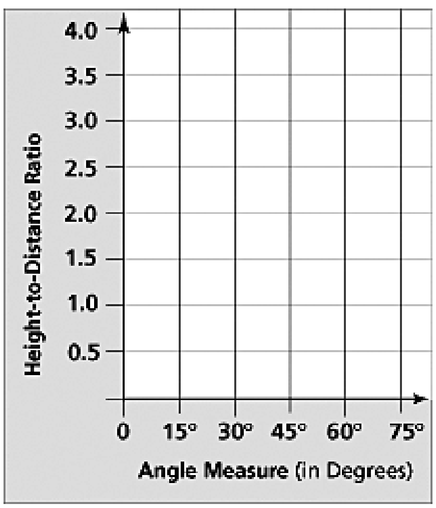
Part C, cont'd.

Problem C9. We've slightly revised the table from Problem C7 to include the 15- and 75-degree angles:

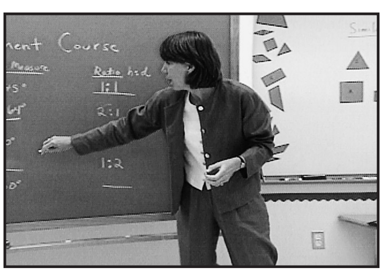
Problem	Measure $\angle\alpha$	Ratio as Decimal
a	15º	0.27
b	45º	1
c	30º	0.58
d	60º	1.72
e	75º	3.73

a. Using the data from this table, plot and connect the points on the Steepness Graph of the height-to-distance ratios for a ladder leaning against a wall at different angles. Use the Ladders Worksheet on page 108 to complete the solution.

Steepness Graph:



b. Examine the information in the Steepness Graph. What happens to $\angle\alpha$ as the *h*:*d* ratio increases?



Video Segment (approximate time: 15:43-18:10): You can find this segment on the session video approximately 15 minutes and 43 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants explore the relationship between the angle of elevation and the height-to-distance ratio. They graph their data to see what happens to the ratio as the angle increases.

Were your findings similar? How would you explain this relationship in your own words?

Problem C10. Suppose that it is safe to be on a ladder when the *h*:*d* ratio is larger than 2 and smaller than 3. Give a range of angles at which the ladder can be positioned safely.

Part C, cont'd.

The Tangent

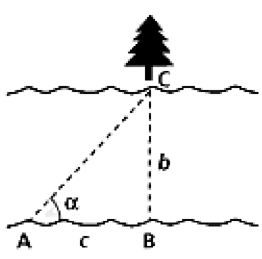
The tangent is the ratio of vertical height to horizontal distance in any context that can be represented with a right triangle. The tangent ratio provides useful information about steepness and can help us determine the measure of $\angle\alpha$, which is often referred to as the angle of elevation. Tables are available that show the relationship between the size of an angle and its tangent. Today, however, most people use the tangent key on a scientific calculator to obtain this information.

Experiment with your calculator, using the data in the table below for verification, to find the following:

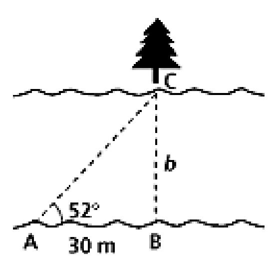
- The corresponding tangent, given an angle
- The corresponding angle, given a tangent **[See Note 8]**

Angle (in Degrees)	Tangent
45	1
46	1.036
47	1.072
48	1.111
49	1.15
50	1.192
51	1.235
52	1.28
53	1.327
54	1.376

We can use right-angle trigonometry to solve problems like those in Part A.



To determine the distance (b) of a tree (point C) across a river, first locate point B directly across the river (where the sighting line is perpendicular to the bank). Next we locate point A at some distance c on the same side of the river as B, and we physically measure the distance between the two (for example, 30 m). Standing at A, we sight C and measure $\angle\alpha$ (using a transit), which we determine to be 52 degrees (pictured at right):



Note 8. If you have not used a scientific calculator recently, first obtain the information in the table given in this section. On some calculators, you enter the angle measure (e.g., 46) and then press the tangent button to get the tangent ratio (e.g., 1.035530314). (Note that the data in the table have been rounded, in this case to 1.036.)

What if you have the h/d tangent ratio and want to find the corresponding angle? In that case, you'd enter the ratio first (e.g., 1.036) and then press the inverse of the tangent key (e.g., tan-1), which is usually a "second" function, to obtain the angle measure (e.g., 46.01298288). This number can also be rounded.

Part C, cont'd.

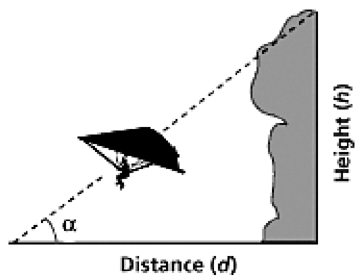
Finally, we set up the tangent ratio to determine the distance between B and C:

tan 52° = *b*/*c*

Using this information, let's calculate the distance *b*.

Problem C11. Find the width of the river at C (distance *b*) when ∠α is 52 degrees. **[See Note 9] [See Tip C11, page 109]**

Other types of problems can also be solved using the tangent. For example, hang gliders are interested in the steepness of their glide paths. The angle that the hang glider makes with the ground as it descends is called a glide angle (∠α in the figure below):



Problem C12. When a hang glider travels a long distance, it is less likely to crash. Three gliders' height-to-distance ratios (sometimes referred to as glide ratios) are given. Sketch the right triangles to show the glide paths. Which glider is the safest?

- Glider 1 — 1:27
- Glider 2 — 0.04
- Glider 3 — 3/78

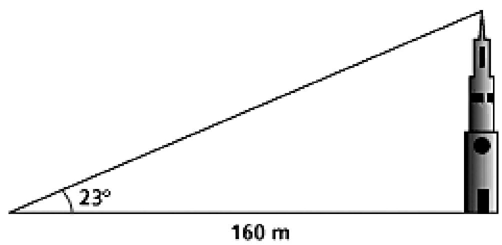
[See Tip C12, page 109]

Problem C13. If the glide angle of a glider is 35 degrees, how much ground distance does a glider cover from a height of 100 m?

Note 9. Notice that this problem is not very different from what you did in Part A. Using a tangent ratio is a kind of shortcut. When you drew similar triangles and set up the proportion between the corresponding sides, you were really calculating the tangent ratio of the triangle with known dimensions, and you then used that to find the unknown length on the other triangle. Now you can calculate the same ratio without the middle step of drawing a similar triangle—you just need to find the tangent of the angle and then use it in the equation.

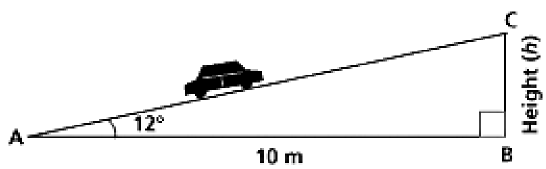
Homework

Problem H1. At a distance of 160 m from a tower, you look up at an angle of 23 degrees and see the top of the tower:



What is the height of the tower?

Problem H2. Compute the height of this extremely steep road at point C for the drawing below:

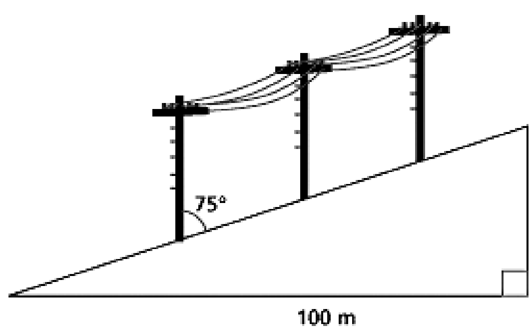


Problem H3. Draw a side view of the flight path for a glider whose glide angle is 5 degrees. What is the glide ratio?

Problem H4. One glider has a glide ratio of 1:40, while another has a glide angle of 3 degrees. Which glider flies farther? Explain why.

Problem H5. Suppose that a glider has a glide ratio of 1:40. It is flying over a village at an altitude of 230 m, and it's 9 km from an airstrip. Can it reach the airstrip? Explain.

Problem H6. An electricity line pole makes an angle of 75 degrees with the road surface, as shown below:

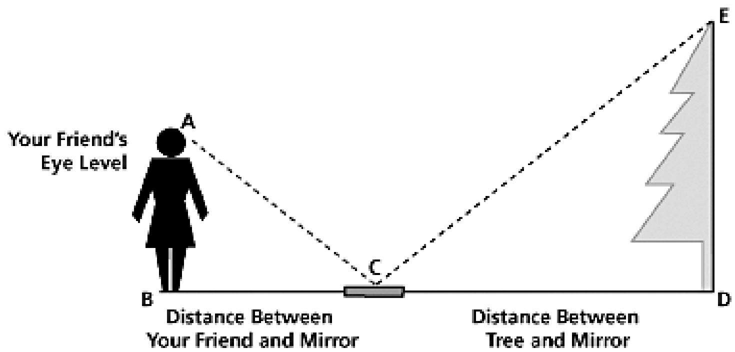


How much does the road rise over a horizontal distance of 100 m?

Homework, cont'd.

Problem H7.

a. Your friend places a mirror 30 ft. from the base of a tall tree. Then she steps back from the mirror until she sees the top of the tree in the mirror's center. What can be said about the angle formed from the treetop to the mirror to the base of the tree, and the angle formed from her head to the mirror to the base of her feet? What do you know about the other angles in the triangles formed below?



b. How might this information be used to determine the height of the tree?

c. You know that your friend is 6 ft. tall and that the mirror is 30 ft. from the base of the tree. After your friend moves back 4 ft. from the mirror, she can see the treetop's reflection. How tall is the tree?

Take It Further

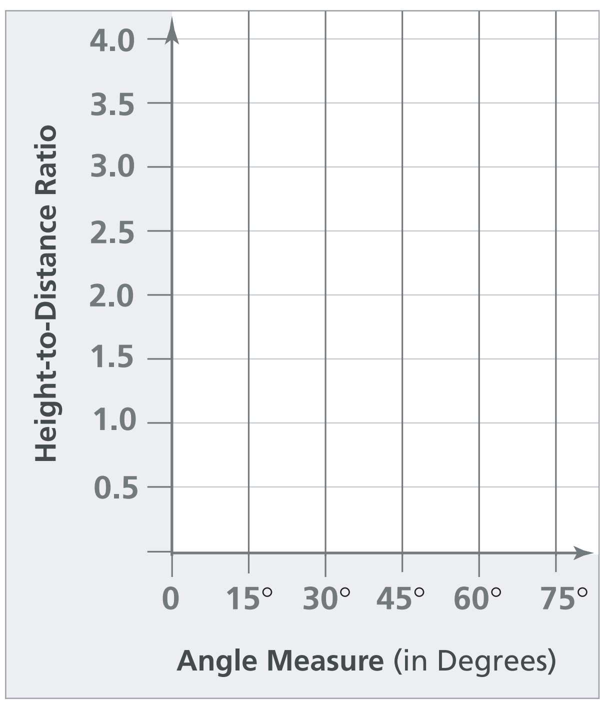
Problem H8. Pretend that you are standing at the equator at noon one day, and the Sun's rays are directly overhead (casting no shadow). Meanwhile, your friend, who is located 787 km away, calls you and tells you that at that very moment the Sun is casting a shadow, and that he had measured and calculated that the Sun's rays are coming in at 7.2 degrees. Knowing that there are 360 degrees around the Earth from its center point, use this information to estimate the Earth's circumference. Compare this estimate of the Earth's circumference to today's known value of 40,075.16 km. **[See Tip H8, page 109]**

Ladders Worksheet

Problems C7-C9

Problem	Measure $\angle \alpha$	*h:d* Ratio	Ratio as Decimal
a			
b			
c			
d			
e			

Steepness Graph:



Tips

Part A: Indirect Measurement With a Transit

Tip A2. Think about how many measurements are needed in order to have a unique triangle.

Part B: Measuring Heights of Tall Objects

Tip B2. Do the sunbeams form an angle? Also, remember that if two of the angles in both triangles are congruent, the triangles must be similar.

Part C: Steepness and Trigonometry

Tip C3. Look for general patterns. For example, what types of ratios result in small angles of elevation? What types result in large angles of elevation? When $h = d$, what is the angle of elevation?

Tip C7. When given $\angle\alpha$, start by drawing the angle. Next, choose an integer (a whole number) as the length for the distance from the wall. Draw the right angle and complete the triangle. Finally, measure the height and determine the *h:d* ratio.

Tip C11. Use algebra to solve the equation tan 52° = *b*/30. First use the trig key on your calculator to find the value of tan 52°, and then multiply both sides by 30.

Tip C12. Rewrite the ratios for Gliders 2 and 3 as unit ratios (1:*x*). Then sketch the right triangles whose sides correspond to the unit ratios for each glider.

Homework

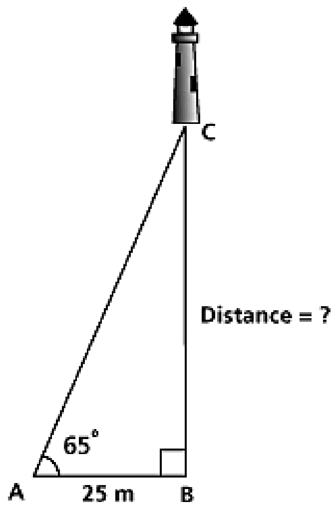
Tip H8. Draw some pictures. The 7.2-degree angle will be opposite the length of 787 km. Fifty of these 7.2-degree angles give a complete circle.

Solutions

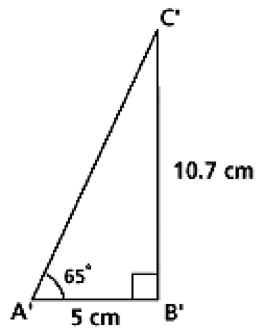
Part A: Indirect Measurement With a Transit

Problem A1. Answers will vary. Here is one example:

a.



b. The similarity ratio here is 5 m:1 cm. Here is the similar triangle:



c. The distance to the lighthouse (the length BC) is 53.5 m.

Problem A2. The three measurements (angle, side, angle) determine a unique triangle; proving that two triangles are similar requires only two more measurements (the two angles in the second triangle). Also, as we've seen in Session 4, every polygon can be divided into triangles, which can be regarded as its basic building blocks. Therefore, triangles will work in every situation, which is why we use them instead of any other polygon.

Problem A3. Using the same triangle, you could find the distance from your other sight point to the tree.

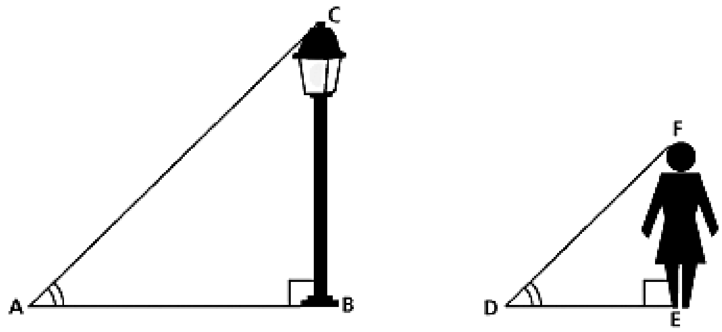
Problem A4. Answers will vary, but here is one example: An indirect measurement can be taken when two figures are known to be similar and when a known measurement is taken from each figure. The ratio of this measurement establishes a scale factor for any other measurements that compare the two similar figures.

Part B: Measuring Heights of Tall Objects

Problem B1. Answers will vary.

Solutions, cont'd.

Problem B2. The triangles are similar because they both have two equal angles. One equal angle is the right angle formed by the lamppost/friend and the flat ground. The other equal angle is the angle formed by the shadow of each object and the Sun's rays (the Sun's rays are parallel lines that strike the ground at the same angle for either shadow).



Problem B3. We can take a known measurement for each object (the length of the shadow) to establish the scale factor by setting up a ratio (AB:DE). Then we multiply the height of the person (EF) by the scale factor to get the height of the lamppost (BC). This is equivalent to setting up a proportion:

AB/DE = BC/EF

Or, using cross multiplication:

BC = (AB • EF)/DE

Problem B4. Answers will vary. You could use the proportion from Problem B3, or alternatively, you could set up a different proportion, which would yield the same result:

AB/BC = DE/EF

Or, using cross multiplication to solve for BC:

BC = (AB • EF)/DE

Both proportions will yield the same result.

Problem B5.

a. It is a derived measure since it is determined by calculations on other measures.

b. Answers will vary depending upon the actual height of the person and the lengths of the shadows. The upper limit will use the maximum lengths for the person and for the lamppost's shadow, and the minimum length for the person's shadow. The lower limit will use the opposites. Assuming that you could know the accurate measure for each of these lengths, the upper limit would be BC = ((AB + 0.25) • (EF + 0.25))/(DE - 0.25). (Each of the amounts AB, EF, DE is the absolute height of the object.)

   The lower limit would be BC = ((AB - 0.25) • (EF - 0.25))/(DE + 0.25). For more information on accuracy, go to Session 2, Part C of this course.

   The best value for the height of the lamppost might be the average of these two limits, since it gives us a reasonable estimate that is close to either limit.

Problem B6. You could use the methods presented in this part to take a derived measure of the height of the tree.

Solutions, cont'd.

Part C: Steepness and Trigonometry

Problem C1.

a. If the ladder is too steep, it may be difficult to climb, and there is a good chance the ladder will fall over backward (say, in a strong wind).

b. If the ladder is not steep enough, it may also be difficult to climb, it is likely to fall forward, and it may not reach high enough to be useful.

Problem C2. Answers will vary. You should find that as the angle between the ground and the ladder increases, the height that the ladder reaches on the wall increases while the distance from the base decreases.

Problem C3. When the angle is 45 degrees, the height and distance are equal. When the angle is larger than 45 degrees, the height-to-distance ratio is greater than 1, and when the angle is smaller than 45 degrees, the ratio is less than 1. Importantly, this ratio is based entirely on the angle, rather than on the length of the actual ladder used.

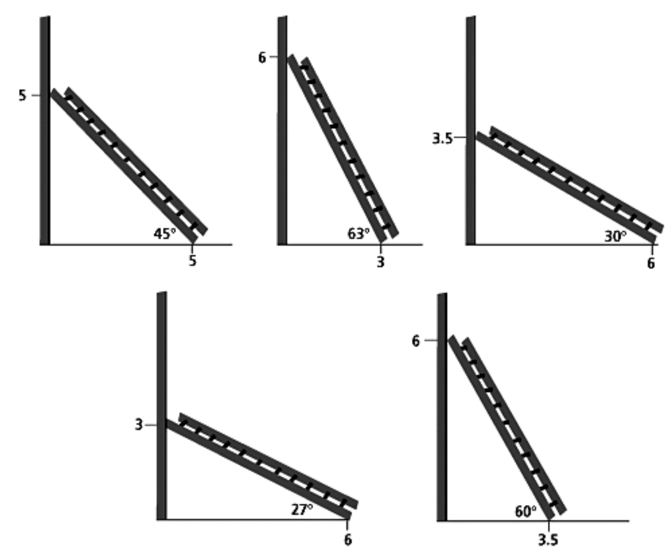
Problem C4. As the ratio increases, the angle increases, but it will always be less than 90 degrees.

Problem C5. By common definition, height is measured along a line that is perpendicular to the base, so the angle must be 90 degrees. If the angle were not 90 degrees, we would not be measuring the vertical height of the ladder against the wall, but some other distance—which would also affect the height-to-distance ratio.

Problem C6.

Ratio (height:distance)	Angle of Elevation	Tangent Notation
3.8/10.3	20°	tan 20° = 0.37
5.5/9.5	30°	tan 30° = 0.58
7.8/7.8	45°	tan 45° = 1
9.5/5.5	60°	tan 60° = 1.73
10.6/2.8	75°	tan 75° = 3.79

Problem C7. Answers may vary. Here are some possibilities:



Solutions, cont'd.

Problem C7, cont'd.

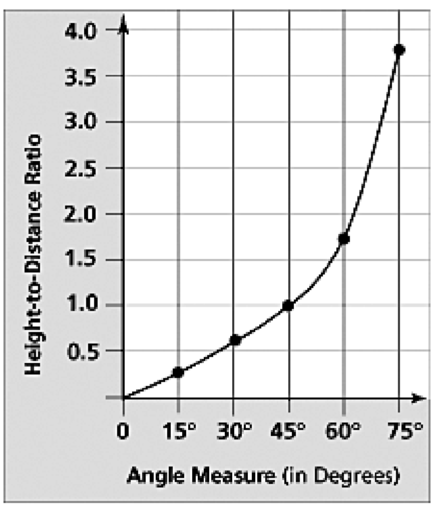
Problem	Measure $\angle\alpha$	h:d Ratio	Ratio as Decimal
a	45°	5:5 (or 1:1)	1
b	63°	6:3 (or 2:1)	2
c	30°	3.5:6	0.58
d	27°	3:6 (or 1:2)	0.5
e	60°	6:3.5	1.72

Problem C8. The ratios are reciprocals (2:1 and 1:2), while the angles are complementary (they sum to 90 degrees). One way to think about this is that if we "reversed" the triangle (switched *h* and *d*), we should also reverse the angles in the triangle. The 90-degree angle remains fixed, so the other two angles will switch. Since they are complementary, if one is 63 degrees, the other is 27 degrees, and vice versa. While it might be easy to see that the measures of both angles sum to 90 degrees, seeing that the *h*:*d* ratios are inverses may not be as obvious.

Notice that the angles in Problem C7 (c) and (e) are 30 and 60 degrees respectively, and are also complementary angles.

Problem C9.

a. Here is the completed Steepness Graph:



b. The *h*:*d* ratio increases as the measure of $\angle\alpha$ increases, but the ratio is increasing at a greater rate. As the angle approaches 90 degrees, the ratio grows increasingly large (with no limit!). Try drawing a triangle with an 85-degree angle and then measure the *h*:*d* ratio. It will be very large! Notice how this is shown on the graph where the curve becomes steeper after the 45-degree mark.

Problem C10. This range can be determined by drawing triangles or by referring to a table of values for these ratios. The smallest safe angle is about 63 degrees (see Problem C7, part b), while the largest is about 72 degrees.

Problem C11. Tan 52° = *b*/30, or 1.28= *b*/30. Multiplying both sides by 30 yields the width of the river, which is 38.4 m.

Solutions, cont'd.

Problem C12. One approach is to rewrite these ratios as unit ratios and then compare them. The ratios are as follows:

Glider 1 — 1:27

Glider 2 — 1:25

Glider 3 — 1:26

Glider 1 has the smallest glide ratio, so it can travel farther (27 m for every 1 m that it descends), and it descends at the slowest rate; therefore, it is the safest.

Another approach is to convert the ratios to decimals and then compare them (this time, looking for the smallest decimal).

Problem C13. Using a calculator, we see that tan 35º = 0.70. Since we know that tan 35º = *h/d*, we just plug in the numbers:

0.70 = 100/*d*

The distance is 100/0.7, or approximately 143 m.

Homework

Problem H1. Using a calculator, we see that tan 23º = 0.42. Therefore, 0.42 = *h*/160, so *h* = 160 • 0.42, or 67.2 m.

Problem H2. Using a calculator, we see that tan 12º = 0.21. Therefore, 0.21 = *h*/10, so *h* = 10 • 0.21, or 2.1 m.

Problem H3. The tangent of a 5-degree angle is 0.0875. This is a glide ratio of about 1:11.4, so the glider flies 11.4 m for every meter it drops.

Problem H4. The tangent of a 3-degree angle is 0.0524. This is a glide ratio of about 1:19, which is much less than 1:40, so the glider with ratio 1:40 flies more than twice as far.

Problem H5. The distance the glider can travel is 230 • 40 = 9,200 m, or 9.2 km. So yes, the glider can reach the airstrip.

Problem H6. The angle of 75 degrees means that the road rises at a 15-degree angle. Tan 15º is about 0.27, which equals *h*/100; therefore, *h* is about 27 m.

Solutions, cont'd.

Problem H7.

a. The two angles are equal. They are known as the angle of incidence and the angle of reflection, and from physics we know they are equal. Also, since we know they are both right triangles, the angle at the top of the tree is equal to the angle at your friend's head.

b. Since we've determined that the angles in the triangles are the same, they are similar triangles. We can set up the following proportion to help us find the height of the tree:

$$\frac{\text{Tree Height}}{\text{Friend's Height}} = \frac{\text{Distance from Tree to Mirror}}{\text{Distance from Friend to Mirror}}$$

c. Using this proportion, we get the following:

   $x/6 = 30/4$

   The height of the tree is 45 ft.

Problem H8. Since 50 of these 7.2-degree angles give a complete circle and we know the length between where the Sun is overhead and where it is at an angle, we can use this to approximate the circumference of the Earth:

$360 \div 7.2 = 50$

$50 \cdot 787 = 39350$ km

It's pretty close! **[See Note 10]**

Note 10. This problem has a great deal of history behind it. Eratosthenes is credited with using this approach to calculate the circumference of the Earth. He concluded that the only explanation for why no shadows fell at Syene at midday on June 21 but did fall at Alexandria was because of the curvature of the Earth. He then devised a method for finding the circumference of the Earth that is based on constant-ratio calculations involving proportions. On the day of the summer solstice, he measured the direction of the Sun's rays as they struck an obelisk in Alexandria and an angle between them. This angle (we'll call it ∠A) can be compared to 360 degrees, and this ratio can in turn be used in the following proportion:

$$\frac{\angle A}{360^\circ} = \frac{\text{Distance Between Alexandria and Syene}}{\text{Circumference of Earth}}$$

He measured ∠A to be 8 degrees and the distance from Alexandria to Syene as 4,800 Greek stadia (a stadia corresponds to approximately 606.75 ft.). He then set up a proportion similar to the one above. Eratosthenes's approximate measurement for the circumference of the Earth was very close to today's modern value.

Notes