

Session 10, Grades 6-8

Classroom Case Studies

This is the final session of the *Measurement* course! In this session, we will examine how measurement concepts from the previous nine sessions might look when applied to situations in your own classroom. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 6–8 begins below. Go to page 195 for grades K–2 and page 205 for grades 3–5.

Key Terms in This Session

Previously Introduced

- area
- surface area
- volume

Introduction

In the previous sessions, we explored many different topics related to measurement. You put yourself in the position of a mathematics learner, both to analyze your individual approach to solving problems and to get some insights into your own understanding of measurement topics. It may have been difficult to separate your thinking as a mathematics learner from your thinking as a mathematics teacher. Not surprisingly, this is often the case! In this session, however, we will shift the focus to your own classroom and to the approaches your students might take with mathematical tasks involving measurement. **[See Note 1]**

For the list of materials that are required and/or optional in this session, **see Note 2.**

Learning Objectives

In this session, you will do the following:

- Explore the development of the concept of area in the middle school grades
- Examine students' understanding of surface area and volume relationships
- Investigate instructional tasks on surface area and volume that are developmentally appropriate for middle school students
- Explore how you might teach related measurement topics

Note 1. This session uses classroom case studies to examine how children in grades 6–8 think about and work with measurement concepts. If you are taking this course on your own, you may want to share your observations on Channel Talk or ask some of your colleagues for their input. Using the classrooms of fellow teachers as well as your own as case studies will allow you to make additional observations.

Note 2. Materials Needed:

- Unit cubes (optional)
- Protractor (optional)

Part A: The Concept of Area (25 min.)

To begin exploring what the teaching of measurement might look like in the classroom, participants in the *Measurement* course first re-examined the big ideas around one topic: area. They considered how students make sense of these ideas and discussed ways to present these concepts to middle school learners.



Video Segment (approximate time: 3:41-5:09): You can find this segment on the session video 3 minutes and 41 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, six teachers discuss some of the important concepts involving area encountered by students in grades 6-8.

Problem A1. Answer the following questions based on your experiences in the course and what you saw in the video:

- What concepts and skills did the teachers mention were important for students to understand?
- Based on your own experiences, what content do students find difficult when studying area?
- What types of activities might be used to help students make sense of these concepts and skills?
- Thinking back to the big ideas of this course, what are some other ideas that students should encounter to help extend and deepen their understanding of the topic of area?

Problem A2. Choose one of the concepts that you listed for Problem A1 and describe an instructional activity that you might use to help students grasp that concept.

Problem A3. In the video segment, Mr. Cellucci and the other teachers discuss the importance of activities in which students actually measure objects, such as going outside to measure shadows and calculate heights. Is it important for older students to engage in measurement activities, or are they able to make sense of the material at an abstract level? In general, what role do manipulative materials play in helping students understand measurement concepts?

Join the Discussion! www.learner.org

Post your answer to Problem A3 on an email discussion list; then read and respond to answers posted by others. Go to the *Measurement* Web site at www.learner.org/learningmath and find Channel Talk.

Part B: Reasoning About Measurement (40 min.)

Exploring Standards

The National Council of Teachers of Mathematics (NCTM, 2000) has identified measurement as a strand in its *Principles and Standards for School Mathematics*. In grades pre-K–12, instructional programs should enable all students to do the following:

- Understand measurable attributes of objects and the units, systems, and processes of measurement
- Apply appropriate techniques, tools, and formulas to determine measurements

In grades 6–8 classrooms, students are expected to do the following:

- Understand both metric and customary systems of measurement
- Understand relationships among units and convert from one unit to another within the same system
- Understand, select, and use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume
- Use common benchmarks to select appropriate methods for estimating measurements
- Select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision
- Develop and use formulas to determine the circumference of circles and the area of triangles, parallelograms, trapezoids, and circles, and develop strategies to find the area of more complex shapes
- Develop strategies to determine the surface area and volume of selected prisms, pyramids, and cylinders
- Solve problems involving scale factors, using ratio and proportion
- Solve simple problems involving rates and derived measurements for such attributes as velocity and density

The NCTM (2000) Measurement Standards suggest that “frequent experiences in measuring surface area and volume can also help students develop sound understandings of the relationships among attributes and of the units appropriate for measuring them. For example, some students may hold the misconception that if the volume of a three-dimensional shape is known, then its surface area can be determined. This misunderstanding appears to come from an incorrect over-generalization of the very special relationship that exists for a cube: If the volume of a cube is known, then its surface area can be uniquely determined. For example, if the volume of a cube is 64 cubic units, then its surface area is 96 square units. But this relationship is not true for rectangular prisms or for other three-dimensional objects in general. . . . Students can reap an additional benefit by considering how the shapes of rectangular prisms with fixed volume are related to their surface area. By observing patterns in the tables they construct for different fixed volumes, students can note that prisms of a given volume that are cube-like (i.e., whose linear dimensions are nearly equal) tend to have less surface area than those that are less cube-like” (NCTM, 2000, pp. 242-243).

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Standards are listed with the permission of the National Council of Teachers of Mathematics (NCTM). NCTM does not endorse the content or validity of these alignments.

Part B, cont'd.

Analyzing a Case Study

To continue the exploration of what measurement topics look like in a classroom at your grade level, you will watch a video segment of a teacher who took the *Measurement* course and then adapted the mathematics to his own teaching situation. We will begin by looking at some of the content addressed in the videotaped lesson. [See Note 3]

Problem B1. In the videotaped lesson, Mr. Cellucci challenges students to construct a rectangular prism with a volume of 72 cm^3 . Students must find a prism with this volume that has the smallest and the largest surface area. Generate a list of dimensions for rectangular solids that have a volume of 72 cm^3 , and then calculate the surface area for each of the different solids. What do you notice about the relationship between the dimensions of a prism and its resulting surface area?

Problem B2. Mr. Cellucci chose the volume of the rectangular prism to be 72 cm^3 as compared to 71 or 73 cm^3 . What might be the purpose of using 72 cm^3 as the volume?



Video Segment (approximate time: 17:26-25:35): You can find this segment on the session video 17 minutes and 26 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

As you watch this video segment, think about how both the lesson and the teacher are assisting students in making sense of the relationships between surface area and volume described in the NCTM Standards.

Problem B3.

- Students use different methods to discover the dimensions that result in a solid with a volume of 72 cm^3 . What problem-solving strategies are students using to find the shape with the smallest surface area?
- Having found the shape with the smallest surface area, the students draw the net of the solid on a white board and build the solid from paper. What are the instructional benefits of examining this solid as a two-dimensional net and a three-dimensional solid?

Problem B4. Mr. Cellucci also asks his students to find the rectangular solid with the largest surface area. How does this task support the NCTM Measurement Standards?

Problem B5. This lesson focuses on the fact that if volume remains constant (in this case, 72 cm^3), the surface area of shapes constructed with that volume can vary. Do you think the mathematical purpose of the lesson is clear? What other factors make a lesson successful?

Join the Discussion! www.learner.org

Post your answer to Problem B5 on an email discussion list; then read and respond to answers posted by others. Go to the *Measurement* Web site at www.learner.org/learningmath and find Channel Talk.

Problem B6. Often we want students to generalize what they have learned. How did Mr. Cellucci use a summary discussion to move students toward generalizing? What generalizations did his students mention?

Note 3. The purpose of the video segments is not to reflect on the teaching style of the teacher portrayed. Instead, look closely at the methods the teacher uses to bring out the ideas of measurement while engaging his students in activities.

Part C: Problems That Illustrate Measurement Reasoning (55 min.)

In this part, you'll look at several problems that are appropriate for students in grades 6–8. For each problem, answer the below questions. If time allows, obtain the necessary materials and solve the problems.

- What is the measurement content in the problem? What are the big ideas that you want students to consider and understand?
- What prior knowledge is required? What later content does it prepare students for?
- How does the content in this problem relate to the mathematical ideas in this course?
- What other questions might extend students' thinking about the problem?
- What other instructional activities or problems might you use in conjunction with this one to further your content goals?

Problem C1. Bicycles are equipped with different types of tires. Twenty-six-inch tires have a diameter of 26 in., whereas 28 in. tires have a diameter of 28 in. You are riding a bicycle with 26 in. tires. If one turn of the pedals moves you forward one tire rotation, how many times must you turn the pedals to ride 1 mile?

Problem C2. Take a unit cube and increase all three dimensions by the scale factor in the table below. For example, to make a new cube that has a scale factor of 2:1, you would double the length, width, and height. The new cube would have dimensions of 2 by 2 by 2, a surface area of 24 square units, and a volume of 8 cubic units. Fill in the chart with the dimensions, surface area, and volume of the new, scaled-up cubes.

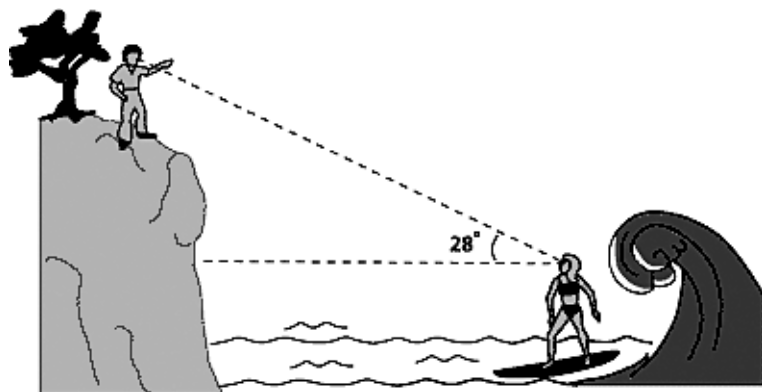
Scale Factor	Dimensions	Surface Area	Volume
1:1			
2:1			
3:1			
4:1			
5:1			
10:1			
25:1			

Examine the surface-area and the volume columns in your table. What patterns of growth do you notice? Can you determine a general rule?

Problem C2 adapted from *Mathscape: Gulliver's Worlds*, p. 31. © 1988 by Glencoe/McGraw Hill. www.glencoe.com/sec/math

Part C, cont'd.

Problem C3. Charlene is out surfing and catches the eye of her friend, Dave, who is standing at the top of a vertical cliff. The angle formed by Charlene's line of sight and the horizontal measures 28 degrees. Charlene is 50 m out from the bottom of the cliff. Charlene and Dave are both 1.7 m tall. The surfboard is level with the base of the cliff. How high is the cliff?



Homework

Solutions are not provided for these homework problems, since answers will vary depending on individual experiences.

Problem H1. Assume that you need to report back to your grade-level team or to the entire school staff at a faculty meeting about your experiences and learning in this course. What are the main messages about the teaching of measurement you would share with your colleagues? Prepare a one-page handout or an overhead or slide that could be distributed or shown at the meeting.

Problem H2. Look at a lesson or activity in your own mathematics program for your grade level that you think has potential for developing students' reasoning about measurement. If you were to use this lesson or activity now, after taking this course, how might you modify or extend it to bring out more of the important concepts about measurement?

Problem C3 adapted from Fendel, D.; Resek, D.; Alper, L.; Fraser, S. *Interactive Mathematics Program Year 1: Shadows*. p. 181. © 1997 by Key Curriculum Press, 1150 65th Street, Emeryville, CA 94608. 1-800-995-MATH. www.keypress.com

Solutions

Part A: The Concept of Area

Problem A1.

- a. The teachers discussed the following ideas that would be important to explore with their students:
 - The dynamic relationship between the perimeter and area of a shape
 - The difference between estimating and physical measuring, which can further be tied to the ideas of accuracy and precision
 - Experiencing area as a physical process of covering a two-dimensional surface in square units
- b. Answers will vary. Students may have difficulty understanding measurement relationships, measurement formulas, indirect measurement, and the ideas of accuracy and precision.
- c. To help students make sense of these concepts and skills, you can help them explore the relationship between area and perimeter of various shapes. They can also estimate the area of irregular shapes. They can see the effect that using different-sized square units has on a measurement and how using smaller units can help make a better approximation.
- d. To deepen and extend their understanding of area, students can explore the effects of a change in dimension, surface area, and volume on the other attributes of a three-dimensional object.

Problem A2. Answers will vary. To deepen students' understanding of area, you can have them examine the effect that changing dimensions will have on the surface area of a rectangular prism. Give students 24 unit cubes and challenge them to make a rectangular solid that has the least possible surface area and one that has greatest possible surface area.

Problem A3. Even older students can benefit from having the experience of physically measuring. Just as number play helps students develop number sense, measuring helps them develop measurement or unit sense. Measurement activities also help students see measurement shortcuts and develop measurement formulas. For example, filling a rectangular box with layers of unit cubes can help students see that a layer is equal to the area of the base, and also that volume can be determined by multiplying the area of the base by the height. The reason for conducting an activity like going outside to measure shadows is that it allows students to see how to apply indirect measurement concepts to the solution of problems. The skills involved relate measurement ideas to proportionality, which is another important topic for middle school students. Hands-on experiences like these help students form a strong conceptual foundation upon which to develop more abstract and complex forms of thinking and analysis.

Solutions, cont'd.

Part B: Reasoning About Measurement

Problem B1.

Rectangular Prisms With a Volume of 72 cm ³	
Dimensions	Surface Area
1 by 1 by 72	290 cm ³
1 by 2 by 36	220 cm ³
1 by 8 by 9	178 cm ³
2 by 2 by 18	152 cm ³
2 by 4 by 9	124 cm ³
3 by 4 by 6	108 cm ³
4 by 4 by 4.5	104 cm ³
4.16 by 4.16 by 4.16	103.8336 cm ³

Volume does not uniquely determine the size of a rectangular prism. In terms of size, as the dimensions of the rectangular prism become more similar, the surface area decreases. Said another way, the surface area of a rectangular prism is minimized as its shape approaches a cube.

Problem B2. Since 72 has many factors (1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72), Mr. Cellucci's choice of 72 cm³ for the rectangular prism's volume allowed for a range of rectangular prisms to be constructed.

Problem B3.

- Some students noticed that rectangular prisms that looked more like cubes had smaller surface areas. Other students focused on the dimensions and saw that as the dimensions "got closer," the surface area "got smaller." One group hypothesized that a perfect cube would minimize surface area, and then set out to determine the necessary dimensions and to construct that cube.
- Drawing the two-dimensional net and three-dimensional solid helped students assign appropriate dimensions and visualize what the rectangular prism would look like before they constructed it. Drawing the net of the three-dimensional solid was also helpful in understanding and determining surface area. Students saw that the total area of the two-dimensional net was the surface area of the rectangular prism and that the net is, in effect, "wrapped around" the solid. By working with the nets, students also focused on the number of identical faces and started to realize they could find the area of certain faces and then multiply to get the total surface area.

Problem B4. One of the standards listed for grades 6-8 is "Develop strategies to determine the surface area and volume of selected prisms,..." which this problem asks students to do (repeatedly). They must determine surface areas, and they must, in a sense, determine volumes in reverse—using a volume and coming up with dimensions of a prism that yield it. In addition, this problem directly addresses the common misconception pointed out in the standards and cited in the previous part, "Exploring Standards."

Problem B5. The mathematical purpose of the lesson is clear. Students understand that they can create a range of rectangular prisms with the same volume. Some of the factors that make the lesson successful include the following:

- The lesson is hands-on. Students get to measure, construct, and fill the prisms with rice.
- Students organized data in a table and looked for patterns. Some students will more readily notice number patterns than geometric patterns.
- Students moved from making two-dimensional nets to three-dimensional solids. Creating nets for the rectangular prisms helped students to visualize them first.

The lesson allowed students to enter into the task at a variety of points. Mr. Cellucci also underscored the practical applications by suggesting situations in which you would want to be "economical" and minimize packaging materials.

Solutions, cont'd.

Problem B6. In the summary discussion, Mr. Cellucci helped students focus on the dimensions, as well as the surface area, of their rectangular prism by prompting them for the dimensions and recording them in a chart. This allowed students to focus on relationships among the dimensions and between dimensions and surface area. The resulting generalizations were the following:

- As the dimensions get closer or the “lengths more similar,” the surface area gets smaller.
- A cube will have the smallest surface area.
- You can find the dimensions of a cube by taking the cube root of the volume. (This was generalized from the students’ understanding of dimensions and area of a square and of taking a square root.)

Part C: Problems That Illustrate Measurement Reasoning

Problem C1.

Solution:

With each rotation, the tire covers the distance of its circumference. So the circumference of one 26 in. tire rotation = 26π = approx. 81.68 in. = approx. 7 ft.

5,280 ft. per mile \div 7 ft. per tire rotation = approx. 754 pedal turns per mile.

Answers to Questions:

- a. The measurement ideas addressed by this problem include determining circumference; solving problems involving ratio and proportion; converting units; and applying appropriate techniques, tools, and formulas to determine measurements.
- b. Working on this problem, students are likely to extend their understanding of circumference as well as ratio and proportion through problem solving. The problem prepares students for rate problems and other algebraic concepts like solving equations.
- c. This problem draws upon the work that was done on exploring circles, in particular the relationship of diameter to circumference. This problem poses a situation where such prior knowledge is applied to solve a practical problem.
- d. You can play with the proportional aspect of this problem by giving students different information, such as distance traveled and number of pedal turns, and asking students to determine tire size. You might also ask students to consider what tire size would be needed in order to reduce the number of pedal turns needed to ride 1 mile by 10% or 25%.
- e. You can make this lesson concrete for students by having them ride a stationary bike, determine the number of tire rotations per pedal and length of rotation, and then answer the question about number of pedal turns per mile. This problem could also serve as a springboard for scale problems involving area of circles.

Solutions, cont'd.

Problem C2.

Solution:

Scale Factor	Dimensions	Surface Area	Volume
1:1	1 by 1 by 1	6 square units (un^2)	1 cubic unit (un^3)
2:1	2 by 2 by 2	$2^2 \cdot 6 = 24 \text{ un}^2$	$2^3 = 8 \text{ un}^3$
3:1	3 by 3 by 3	$3^2 \cdot 6 = 54 \text{ un}^2$	$3^3 = 27 \text{ un}^3$
4:1	4 by 4 by 4	$4^2 \cdot 6 = 96 \text{ un}^2$	$4^3 = 64 \text{ un}^3$
5:1	5 by 5 by 5	$5^2 \cdot 6 = 150 \text{ un}^2$	$5^3 = 125 \text{ un}^3$
10:1	10 by 10 by 10	$10^2 \cdot 6 = 600 \text{ un}^2$	$10^3 = 1,000 \text{ un}^3$
25:1	25 by 25 by 25	$25^2 \cdot 6 = 3,750 \text{ un}^2$	$25^3 = 15,625 \text{ un}^3$

The surface area of a cube is increased by the scale factor squared. The volume is increased by the scale factor cubed.

Answers to Questions:

- Students working on this problem are likely to deepen their understanding of measurable attributes of a cube (dimension length, surface area, and volume) as well as the relationship between the attributes. The big idea for students to consider is how the surface area and volume grow when a cube is increased (or decreased) by a certain scale factor.
- The problem extends and deepens understanding of surface area and volume of cubes, perfect squares, cubed numbers, and patterns. It prepares students for continued work with exponents.
- This problem also draws on the work that was done in Session 8. In that session, the participants explored how each of the dimensions change based on the scale factor. As a result, this will affect how surface area and volume change for scaled cubes.
- Have students use manipulatives to build the cube and determine the dimensions, surface area, and volume. Explore how the volume also changes. If you know the volume of a cube, can you determine the dimensions and surface area? Why?
- Students can begin to examine and compare the surface area-to-volume ratio for both rectangular prisms and cubes. They can also make comparisons with other shapes or objects, and explore which of them will have a maximum or minimum surface area-to-volume ratio.

Solutions, cont'd.

Problem C3.

Solution:

Since Dave and Charlene are the same height, the 28-degree angle measures exactly the height of the cliff. Visualize closing off the angle to form a triangle and then sliding that triangle down to water level. The side opposite the 28-degree angle would align exactly with the cliff. Use similar triangles to determine the height of the cliff. Using a protractor, draw a right triangle with a 28-degree angle opposite the vertical leg forming the right angle. Measure the length of the two legs of the right triangle with a ruler. Then set up a proportion between those two sides and the height of the cliff and 50 m length in the surfing triangle. The solution should give you a cliff height of roughly 26.5 m.

Answers to Questions:

- a. The measurement ideas in this problem include using appropriate techniques to accurately measure angles and lengths and using ratio and proportion to determine length. Students will use similar triangles. Note: Some reasoning skills are necessary to understand that knowing that Charlene and Dave's heights are equal allows you to find the height of the cliff.
- b. This problem builds on prior work with protractors as well as proportional reasoning. The problem provides an opportunity to apply knowledge of similar triangles.
- c. In the course, teachers explored similar triangles to solve similar problems.
- d. Students could be given the height of the cliff and then asked to determine the distance Charlene is from the cliff. They could also be asked to think about how changing the angle formed by the horizontal side and Charlene's line of sight affects the height of the cliff.
- e. This problem can be followed with a series of "shadow" problems in which students have to determine the height of trees, buildings, or flagpoles by measuring the shadow of a person looking at the top of the structure.

Notes
