

# 3

## Addition and Subtraction

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In order to use addition and subtraction effectively, children must first attach meaning to these operations. One way for young children to do this is by manipulating concrete objects and connecting their actions to symbols. However, this is not the only way. They extend their understanding of situations involving addition and subtraction by solving word problems.

In this chapter we investigate how children solve addition and subtraction word problems involving small quantities and address several questions: How do children decide which operation is called for? (Is it an addition or subtraction situation?) How do children represent the mathematics symbolically? (Can they write an appropriate number sentence or equation to represent the situation?) How do children think numerically to perform the needed computation?

### 1. Types of Addition and Subtraction Word Problems

Students extend their understanding of and skill with addition and subtraction by solving word problems based on different meanings or interpretations of these operations. For example, take these two subtraction problems:

*I have seven apples and Carla has four apples. How many more apples do I have than Carla?*

*I had seven apples and ate four apples. How many apples are left?*

Both situations can be expressed with the same number sentence:  $7 - 4 = \square$ . However, the first problem requires a “comparison” interpretation, the second, a “take away” interpretation. When an operation is reduced to symbols, it is impossible to determine which meaning or interpretation is being represented. So that students can learn that there are multiple interpretations of an operation and expand their repertoire of situations that model the operation, we need to give them a variety of problems to solve.

Evidence suggests that the general meaning of a problem rather than specific words or phrases determines both the difficulty of the problem and the processes students use to solve it. In other words, it is how the operation is expressed and, by extension, a student’s ability to make sense of that meaning rather than grammatical considerations such as the sequence of information and the presence of cue words that make problems easier or harder. In general, difficulties with word problems do

not occur because students cannot read the words but because they cannot make sense of the mathematical relationships expressed by these words. Students' understanding of the different kinds of relationships in word problems is improved by solving and discussing problems.

A common classification scheme identifies four broad categories of addition and subtraction based on the type of action or relationship in the problems: *join*, *separate*, *part-part-whole*, and *compare* (Carpenter et al. 1994, 1999). Within these four broad categories, there are a total of eleven problem types. (Many of these problems use the same key words, even though their structure is different.) Often, textbooks include only one or two types of addition and subtraction problems. However, to become a proficient and competent user of mathematics, students need to be able to solve all types of problems.

### *Join Problems and Separate Problems*

Join problems and separate problems involve actions that increase or decrease a quantity, respectively. In both categories, the change occurs over time. There is an initial quantity that is changed either by adding something to it or by removing something from it, resulting in a larger or smaller final quantity. There are three subsets of each of these problem types, depending on which quantity the solver is being asked to determine: the result, the amount of change, or the initial quantity.

JOIN	EXAMPLE PROBLEM
Result Unknown	Laina had four dolls. She bought two more. How many dolls does she have now? $4 + 2 = \square$
Change Unknown	Laina had four dolls. She bought some more dolls. Now she has six dolls. How many dolls did Laina buy? $4 + \square = 6$
Initial Quantity Unknown	Laina had some dolls. She bought two more dolls. Now she has six dolls. How many dolls did Laina have before she bought some more? $\square + 2 = 6$
SEPARATE	EXAMPLE PROBLEM
Result Unknown	Rodney had ten cookies. He ate three cookies. How many cookies does Rodney have left? $10 - 3 = \square$
Change Unknown	Rodney had ten cookies. He ate some of the cookies. Now he has seven cookies left. How many cookies did Rodney eat? $10 - \square = 7$
Initial Quantity Unknown	Rodney had some cookies. He ate three cookies. Now he has seven cookies left. How many cookies did Rodney have to start with? $\square - 3 = 7$

In join problems and separate problems, the action of adding or subtracting is explicit. You can help students solve both types of problems by asking them to model these actions with objects. For example, using the join/amount-of-change-unknown problem above (*Laina had four dolls. She bought some more dolls. Now she has six dolls. How many dolls did Laina buy?*), students first can represent the four dolls using four blocks. Mimicking the action in the problem, students add some more blocks (dolls) until there are a total of six blocks. By then asking your students to explain how they determined the number of blocks to add, you help them form mental models of this type of situation as well. After solving many problems of this type, students eventually no longer need the physical model and can deal with the relationships symbolically.

Which quantity in a problem is unknown contributes to the overall difficulty of the problem. In general, when the result is unknown, students are more likely to be able to make sense of the relationships. Students are most familiar with result-unknown problems, because they encounter many similar problems in their everyday lives. These types of problem are also more heavily represented in textbooks.

Join problems and separate problems in which the initial quantity or the amount of change is unknown are more difficult for students. One reason is that these problems are often presented in language that suggests one action (e.g., separating) but require using the opposite action (e.g., joining) to find the answer. For example, take the following separate problem in which the initial quantity is unknown: *Rodney had some cookies. He ate three cookies. Now he has seven cookies left. How many cookies did Rodney have to start with?* There is a separating action in the problem, but a child who knows that  $\square - 3 = 7$  describes the problem can solve it using addition, or by counting on from seven: “Eight, nine, ten. Rodney started with ten cookies.” Similarly, some join problems are solved by subtracting. It’s important to notice whether a child is able to recognize the operation that matches the situation, can represent the number sentence or equation correctly, and then can think numerically to find the answer. It is useful for teachers to talk with students about each aspect of problem solving: *What is the problem describing? How can you write that down? How can you find the answer?*

### **Part-Part-Whole Problems**

Part-part-whole problems do not use action verbs—action neither occurs nor is implied. Instead, relationships between a particular whole and its two separate parts are established. There are two types of part-part-whole problems. In one type, the sizes of both parts are given and the student is asked to find the size of the whole. In the other type, the size of one part and the size of the whole are provided and the student is asked to find the size of the other part.

PART-PART-WHOLE	EXAMPLE PROBLEM
Whole Unknown	Five boys and three girls are on the basketball team. How many children are on the basketball team? $5 + 3 = \square$
One Part Unknown	Eight children are on the basketball team. Five are boys and the rest are girls. How many girls are on the basketball team? $5 + \square = 8$

Part-part-whole problems involve a comparison of “parts” (subsets) to the “whole” (set). While the part-part-whole problems above could be modeled with manipulatives, the language in the problems does not suggest any action joining the “parts,” or subsets. This static relationship between subsets is the subtle difference that distinguishes these problems from join problems involving action.

### Compare Problems

Compare problems involve a comparison of two distinct, unconnected sets. Like part-part-whole problems, compare problems do not involve action. However, they differ from part-part-whole problems in that the relationship is not between sets and subsets. There are three types of compare problems, depending on which quantity is unknown: the difference (the quantity by which the larger set exceeds the smaller set), the quantity in the larger set, or the quantity in the smaller set. A relationship of *difference*, *more than*, or *less than* is found in compare problems.

COMPARE	EXAMPLE PROBLEM
Difference Unknown	Ahmed has two brothers. Christine has three brothers. Christine has how many more brothers than Ahmed? $3 - 2 = \square$ or $2 + \square = 3$
Larger Quantity Unknown	Ahmed has two brothers. Christine has one more brother than Ahmed. How many brothers does Christine have? $2 + 1 = \square$
Smaller Quantity Unknown	Christine has one more brother than Ahmed. Christine has three brothers. How many brothers does Ahmed have? $\square + 1 = 3$ or $3 - \square = 1$

Compare and part-part-whole problems exemplify that the operations of addition and subtraction are based on the relationships between sets and subsets. One reason to ask students to solve a variety of problem types is so that they will generalize the meaning of these operations beyond “actions” to relationships between sets.

Students’ ability to solve the various problem categories is related to their ability to recognize the distinctions among them. Contexts and wording that indicate the actions or relationships in a problem can make a problem easier or more difficult. For example, consider these two problems:

*There are five boys and eight hats. How many more hats than boys are there?*

*There are five boys and eight hats. If each boy puts on a hat, how many hats are left over?*

Carpenter and his colleagues (1994, 10) found that the second problem is easier for students because the action is more explicit. The magnitude of the numbers in a problem also affects its level of difficulty. The following problems are both separate

problems in which the change is unknown, but the second one is more difficult for young students because the numbers are larger:

*Liz had twelve pennies. She gave some pennies to Caitlin. Now she has eight pennies. How many pennies did Liz give Caitlin?*

*Liz had 45 pennies. She gave some pennies to Caitlin. Now she has 28 pennies. How many pennies did Liz give Caitlin?*

However, after students solve many separate/amount-of-change-unknown problems that involve small quantities, they are more likely to be able to answer similar problems with larger quantities.

## Activity



### Classifying Addition and Subtraction Word Problems

Before reading further, discuss with a colleague the four types of addition and subtraction problems. Next classify each of the following problems as join, separate, part-part-whole, or compare. Indicate which quantity is unknown and write a number sentence that represents the relationships expressed in each problem.

1. Carlton had three model cars. His father gave him four more. How many model cars does Carlton have now?
2. Juan has nine marbles. Mary has six marbles. How many more marbles does Juan have than Mary?
3. Janice has three stickers on her lunch box and four stickers on her book bag. How many stickers does she have in all?
4. Catherine had a bag of four gummy bears. Mike gave her some more. Now Catherine has seven gummy bears. How many gummy bears did Mike give her?
5. A third grader has seven textbooks. Four textbooks are in his desk. The rest of his textbooks are in his locker. How many textbooks are in his locker?
6. Vladimir had some baseball cards. Chris gave him 12 more. Now Vladimir has 49 baseball cards. How many baseball cards did Vladimir have before he received some from Chris?
7. Keisha had some crayons. She gave two crayons to Tanya. Now Keisha has nine crayons. How many crayons did Keisha have in the beginning?
8. Anthony had nine library books on his bookshelf. He returned six books to the library. How many books are left on his bookshelf?
9. There are nine board games in Joyce's room. Mariah has six fewer board games than Joyce. How many board games does Mariah have?
10. Eric weighed 200 pounds. During the summer, he lost some weight. Now he weighs 180 pounds. How many pounds did Eric lose?
11. Eli had some money. He gave his brother Johannes \$5.50. Now Eli has \$18.50 left. How much money did Eli have to begin with?
12. Graziella has four CDs. Fadia has eight more CDs than Graziella. How many CDs does Fadia have?

### Things to Think About

Sometimes it's difficult to distinguish part-part-whole problems from join problems and separate problems. The main difference is that in part-part-whole problems the relationship between entities is static (as in question 3 above), whereas

the relationships in join problems or separate problems are always described using combining or separating action verbs. The different addition and subtraction problems are not equally easy (or difficult) for students. In general, students find result-unknown problems in the join and separate categories, whole-unknown problems in the part-part-whole category, and difference-unknown problems in the compare category easier than problems in the remaining categories. Furthermore, the types of quantities used in problems can make them easier or more difficult; crayons are easy to model and count, whereas pounds are a more abstract quantity to represent.

Some of the problems (1 and 3, 8 and 9) can be represented by the same number sentence. You might want to write a few number sentences and then make up different types of problems that fit each sentence. This is another way to extend your own understanding of the different types of problems.

Three of the problems in this activity (1, 4, and 6) are join problems. In number 1, the result is unknown ( $3 + 4 = \square$ ), in number 4 the amount of change is unknown ( $4 + \square = 7$ ), and in number 6 the initial quantity is unknown ( $\square + 12 = 49$ ). There are four separate problems: 7, 8, 10, and 11. In number 8, the result is unknown ( $9 - 6 = \square$ ); in number 10, the amount of change is unknown ( $200 - \square = 180$ ); and in numbers 7 and 11, the initial quantity is unknown ( $\square - 2 = 9$  and  $\square - 5.50 = 18.50$ ). The two part-part-whole problems are 3 and 5. In number 3, the whole is unknown ( $3 + 4 = \square$ ), and in number 5, one of the parts is unknown ( $4 + \square = 7$ ). Finally, problems 2, 9, and 12 are compare problems. In number 2, the difference is unknown ( $9 - 6 = \square$ ); in number 9, the smaller quantity is unknown ( $9 - \square = 6$  or  $\square + 6 = 9$ ); and in number 12, the larger quantity is unknown ( $4 + 8 = \square$ ). ▲

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## 2. Solution Strategies

Young children use informal knowledge and different types of strategies to solve word problems and perform computations (Carpenter et al. 1994, 1999). Some of the strategies that students use are based on the structure and semantics of the problem. Other strategies are related to students' growing sense of number and their understanding of mathematical properties and place value. In order to plan instruction that helps students develop increasingly sophisticated and efficient problem-solving strategies, it is important to recognize a variety of strategies and to understand how students' strategies develop.

Students tend to use three types of strategies when solving simple addition and subtraction word problems:

1. Strategies based on direct modeling with fingers or physical objects.
2. Strategies based on counting sequences.
3. Strategies based on number sense.

These strategies are hierarchical: students progress from modeling to counting to using number sense. However, this does not mean that students use only one strategy type for all problems; students apply different types of strategies to different types of problems. For example, when solving a separate word problem in which the amount of change is unknown, a student might use a modeling strategy. When solving a separate problem in which the result is unknown, the same student might use a counting strategy. And when solving a join problem involving numbers less

than five in which the result is unknown, this student might use her knowledge of number facts, although she might revert to using less sophisticated strategies in join problems involving larger numbers.

### ***Modeling Strategies***

In modeling strategies, students use physical objects such as blocks, counters, and fingers to model the actions and/or relationships in a problem. They then count some or all of these objects to obtain an answer. Many young students use modeling strategies when they first start to solve addition and subtraction problems. Older students unfamiliar with a particular problem type often use a modeling strategy to make sense of the problem. There are five common modeling strategies: joining all, separating from, separating to, adding on, and matching.

***Joining all.*** In this addition strategy, students use physical objects to represent each of the addends in a problem. The answer to the problem is found by joining the sets of objects and counting them all, starting with one. Sometimes students first join the sets and then count all the items, sometimes they count one set followed by the other set. Interestingly, students don't seem to differentiate whether or not they physically join the sets, as long as they have modeled each set.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?* Using objects or fingers, the student makes a set of 2 objects and a set of 5 objects. Then he counts the union of the two sets, starting with one.

***Separating from.*** In this subtraction strategy, students use concrete objects to model the action of separating out the smaller quantity given in the problem. Usually the student counts the remaining objects to arrive at the answer.

*There were seven boys playing tag. Two boys went home. How many boys were still playing?* Using objects or fingers, the student makes a set of 7 objects. She removes 2 objects. The number of remaining objects is the answer.

***Separating to.*** This subtraction strategy is similar to the separating-from strategy except that objects are removed from the larger set until the number of objects remaining is equal to the smaller number given in the problem. Counting the number of objects removed provides the answer. This strategy involves some trial and error in that a student has to keep checking to see whether the appropriate amount still remains. Students often use this strategy to solve separate problems in which the change is unknown.

*There were 7 boys playing tag. Some went home. Now there are 2 boys playing tag. How many boys went home?* The student counts out a set of 7 objects. Then he removes objects until only 2 remain. The number of objects removed is the answer.

***Adding on.*** This strategy involves an addition action and is used by students to solve both addition and subtraction problems. A student sets out the number of objects equal to the smaller given number (an addend) and then adds objects one at a time until the new collection is equal to the larger given number. Counting the number of objects "added on" gives the answer. This strategy also involves some trial and error in that a student has to check regularly to see whether the larger number has been reached.

*Liz had two apples. Lyman gave her some more. Now Liz has five apples. How many apples did Lyman give her?* The student makes a set of 2 objects. Then she adds objects to the set one at a time until there is a total of 5 objects. She finds the answer by counting the number of objects added.

**Matching.** This concrete strategy is used by many students to solve comparison problems in which the difference is unknown. The student puts out two sets of objects, each set representing one of the given numbers. The sets are matched one to one. Counting the objects without matches gives the answer.

*Tom has five brothers. Juan has two brothers. How many more brothers does Tom have than Juan?* The student creates a sets of 5 objects and a set of 2 objects. He matches the objects in each set one to one and counts the number of unmatched objects.

### Counting Strategies

Counting strategies are more advanced solution processes than modeling strategies, because they are more abstract and there is more flexibility in which one to choose. The shift from modeling to counting depends on the development of certain number concepts and counting skills. Students must understand the relationship between counting and the number of elements in a given mathematical set (*cardinality*), must be able to begin counting at any number, and must be able to count backward. For some situations, students must also be able to keep track of how many numbers they have counted and at the same time recognize when they have reached the appropriate number. Activities that focus on these counting skills and relationships will further students' ability to apply counting strategies to the solution of problems.

There are six common counting strategies: counting all, counting on from first, counting on from larger, counting down from, counting down to, and counting up from given (this list isn't exhaustive—don't be surprised if your students invent new ones!). In helping students use these strategies, it is important to realize that counting strategies are not mechanical techniques that students can simply memorize. Counting strategies are conceptually based and build directly on modeling strategies. Thus students need opportunities to connect modeling strategies with counting strategies. Many counting strategies have direct links to specific modeling strategies:

MODELING STRATEGIES	COUNTING STRATEGIES
Joining all	Counting all
Separating from	Counting down from
Separating to	Counting down to
Adding on	Counting up from given
Matching	None
None	Counting on from first
None	Counting on from larger



One way to help young students link strategies is to have them discuss how they got their solution. After students have presented their processes, teachers can highlight the similarities between two related strategies (e.g., separating from and counting down from). Furthermore, we can encourage the use of counting strategies and provide opportunities for students to practice them.

**Counting all.** This addition strategy is similar to the joining-all modeling strategy except that physical models or fingers are not used to represent the addends. As the name implies, students start the counting sequence with one and continue until the answer is reached. This strategy requires that students have a method of keeping track of the number of counting steps in order to know when to stop. Most students use their fingers to keep track of the number of counts. (Fingers here play a different role than in the joining-all modeling strategy; they are used to keep track of the number of steps rather than to model one of the addends.)

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?* The student begins the counting sequence with 1 for two counts (1, 2) and then continues on for 5 more counts (3, 4, 5, 6, 7). The answer is the last term in the counting sequence.

**Counting on from first.** With this addition strategy, the student recognizes that it is not necessary to reconstruct the entire counting sequence and begins “counting on” from the first addend in the problem.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?* The student begins the counting sequence at 2 and continues on for 5 counts. The answer is the final number in the counting sequence.

**Counting on from larger.** This addition strategy is identical to the counting-on-from-first strategy except that counting begins from the larger of the two addends. This is a more sophisticated counting-on strategy, since implicit in its application is that the student understands that the order of the addends does not matter in addition problems.

*Katie had two stickers. She bought five more stickers. How many stickers does Katie have now?* The student begins the counting sequence at 5 and continues on for 2 counts. The answer is the final number in the counting sequence.

**Counting down from.** This subtraction strategy is the parallel counting strategy to the separating-from modeling strategy. In this strategy students initiate a backward counting sequence beginning at the given larger number. The counting sequence contains as many numbers as the given smaller number.

*There were seven boys playing tag. Two boys went home. How many boys were still playing?* The student begins a backward counting sequence at 7. She continues the sequence for 2 counts (i.e., 6, 5). The last number in the counting sequence (5) is the answer.

**Counting down to.** This subtraction counting strategy is parallel to the separating-to modeling strategy. Students use a backward counting sequence until the

smaller number is reached. How many numbers there are in the counting sequence is the solution. Students often use their finger to keep track of the counts, but they are not actually modeling the situation.

*There were seven children playing tag. Some went home. Now there are two children playing tag. How many children went home?* The student starts a backward counting sequence at 7 and continues until 2 is reached (i.e., 6, 5, 4, 3, 2). The answer is how many numbers there are in the counting sequence (5).

**Counting up from given.** This counting strategy is parallel to the adding-on modeling strategy. The student initiates a forward counting strategy from the smaller number given. The sequence ends with the larger number given. The student keeps track (often using his or her fingers) of how many numbers there are in sequence.

*Liz had two apples. Lyman gave her some more. Now Liz has five apples. How many apples did Lyman give her?* The student starts counting at 2 and continues until 5 is reached (i.e., 3, 4, 5). The answer is how many numbers there are in the sequence (3).

## Activity



### Matching Problems and Strategies

Understanding the relationships within problems will help you link problem structure to students' solution strategies. Analyze each problem in activity 1 in terms of which strategies could be used to model the actions or relationships in the problem. Indicate both a modeling strategy (with objects) and a counting strategy. Some problems can be solved using a number of different strategies or by using a strategy not presented here.

#### **Things to Think About**

Students' strategy choices are influenced by a number of factors. First, students are most likely to pick a solution strategy that matches the structure of a problem. Whether the strategy chosen is a modeling, counting, or number sense strategy depends in part on students' familiarity with the type of problem (can they make sense of which operation to use?) and on students' understanding of number and counting. When students have not made sense of the relationships in a problem and have not yet connected these relationships to specific operations, they are more likely to use a modeling strategy.

Problems are much more difficult if students do not have a process available to model the actions or relationships. For example, a compare problem in which the difference is unknown is difficult if students have never considered or seen a matching strategy. You can help students learn new strategies by creating specific problems for them to solve and then having them discuss their various solution strategies in pairs and as a whole class. New approaches and strategies are often introduced this way. While occasionally you may wish to model a solution strategy for students, it is important that students don't just observe the strategy but use and discuss it. Students also need many opportunities to apply new strategies. After a strategy has been introduced, you should assign problems that enable students to practice and refine the particular strategy.

Some strategies are not as widely used as others (counting down is not used

as much as counting up from given, for example), and some students never use some of the strategies. In many cases, students don't even differentiate between strategies (some look at counting on and counting up from given as the same strategy). Likewise, as students mature, they often change which strategies they use (the matching strategy is abandoned after the early grades, for example).

Here are the most likely ways to solve the problems in activity 1:

1. Counting all; counting on from larger.
  2. Matching; counting up from given.
  3. Counting all; counting on from larger.
  4. Adding on; counting up from given.
  5. Separating from; counting down from.
  6. Adding on; counting up.
  7. Counting all; counting on from larger.
  8. Separating from; counting down from.
  9. Separating from; counting down from.
  10. Separating to; counting down to.
  11. Counting all; counting on from larger.
  12. Separating from; counting down from. ▲
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### *Number Sense Strategies*

Students eventually replace modeling and counting strategies with number sense strategies. To do so (1) they must understand whether the relationships and actions in a problem require them to add or subtract, and (2) they must be able either to recall number facts or to use known number facts to derive new facts.

The relationship between counting and mental strategies is not clear. We do know that students' previous use of counting strategies helps them recall number facts. Certainly, the ability to move to this level of abstraction depends in part on being able to make sense of the relationships in many types of problem, and this means students must have solved many different kinds of problems.

Being able to use known number facts to derive new facts is linked to an understanding of part-whole relationships (not to be confused with the part-part-whole problem type described in section 1). Quantities can be interpreted as comprising other numbers. For example, a set of eight objects can be represented using two or more parts such as 0 and 8; 1 and 7; 2, 2, and 4; eight 1s; and so on. Eight is the "whole," the numbers making up eight are the "parts." If students do not understand number in terms of part-whole relationships, they are not able to decompose wholes and recombine parts flexibly. For example, to find the answer to  $6 + 8$ , some students decompose 8 into two parts (6 and 2) and reinterpret the calculation as  $6 + 6 + 2$ . If they already know that  $6 + 6 = 12$ , they can use that information to determine that  $6 + 8 = 14$ . Some researchers believe that the most important conceptual achievement in the early grades occurs when students interpret number in terms of part-whole relationships. Interestingly, the use of part-whole reasoning to find the answers to basic facts is not limited to superior students.

Students' solutions involving part-whole relationships are often based on number facts that sum to 10. For example, a student might calculate  $4 + 7$  like this: *Three plus seven is ten, and four plus seven is just one more, so the answer is eleven.* This student knows that the number fact for  $3 + 7$  is 10 and understands that 4 can be thought of

as 3 and 1 even though that relationship is not explicitly stated. Doubles facts are also used by many students to derive new facts. For example, to solve  $7 + 8$  a student might respond: *Eight plus eight is sixteen, and seven plus eight is just one less than sixteen, so the answer is fifteen.*

Other factors that contribute to students' ability to use mental strategies based on number sense are their knowledge and understanding of addition and subtraction properties and relationships. Knowing about the commutative property, which states that the order of the addends (e.g.,  $9 + 4$  or  $4 + 9$ ) does not affect the sum, lets students transpose a problem to an easier form. The inverse relationship between addition and subtraction can also help students find answers (e.g., knowing that  $6 + 7 = 13$ , it follows that  $13 - 6$  is 7) or develop new strategies (e.g., counting up).

The time needed before students consistently use mental strategies to solve problems varies, but for some students it can take a number of years. Mathematical tasks that highlight part-whole relationships in general and that focus on doubling and numbers summing to ten in particular provide a foundation on which students can build. Likewise, instruction that highlights the relationship between addition and subtraction and enables students to reflect on properties also contributes to children's number and operation sense.

## Activity

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### Observing How Students Solve Problems

Write one or two problems that fit the classification scheme described in section 1 and are appropriate for the students you teach. (You may want to begin with join problems in which the result is unknown and separate problems in which the amount of change is unknown.) In individual five-minute interviews, ask students to solve the problems. Record each student's solution strategy for each problem. When you have all your data, group students' solution strategies by type. How many students are using modeling, counting, or number sense strategies? How might you use this information to further students' understanding?

### Things to Think About

When watching students solve problems, it isn't always easy to determine what strategy they are using, since they may be unable to articulate their thinking. You may have to ask probing questions or watch for overt behavior in order to hypothesize about students' thought processes. Furthermore, when a child uses his fingers, is he using a modeling strategy or a counting strategy? It's a modeling strategy if the fingers are used to show the quantities in the problem; it's a counting strategy if the fingers are used to keep track of how many numbers are in a counting sequence.

You may find that students use different strategies for join problems in which the result is unknown than they do for separate problems in which the amount of change is unknown. This may be partly because they haven't had much experience with the latter type of problem.

If students are using modeling strategies, you may wish to include mathematical tasks and sequences in your instruction that support the development of counting strategies. Providing students with many opportunities to practice counting forward and backward (especially starting in the middle of a sequence),

identify patterns when counting (e.g., *When we count by 100s, what changes with each increase? What doesn't change with each increase?*), and connect a modeling strategy with a counting strategy will help. If students are using any of the counting strategies, you can focus on instruction that supports the use of more sophisticated methods. Activities that enable students to grapple with part-whole relationships; that emphasize number combination strategies such as doubles, doubles plus one, and doubles minus one (e.g.,  $5 + 6$  can be thought of as  $5 + 5 + 1$  or as  $6 + 6 - 1$ ); and that have students exploring the relationship between addition and subtraction will all contribute to your goal. ▲

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## Teaching Addition and Subtraction

Students' understanding of addition and subtraction forms the foundation for more advanced work with these operations in upper elementary and middle school. As they progress through the grades, students learn a variety of methods to perform multidigit computations quickly and easily. They learn to solve problems that involve multiple steps, rational numbers, integers, and unnecessary data. Many of their initial strategies are abandoned as their ability to deal with symbols and generalizations increase. However, students often revert to using simpler strategies when they are presented with problems that contain larger numbers, decimals, or fractions.

# 4

## Multiplication and Division

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Multiplication and division have always been important topics in elementary school mathematics. However, in the past, instruction has focused primarily on helping students develop procedural competency with basic facts and paper-and-pencil algorithms. This procedural competency is an important goal (and suggestions for helping students become competent with multiplication and division facts and algorithms are included in chapter 2), but recent research makes it clear that students also need to develop deep conceptual knowledge of multiplication and division. The National Council of Teachers of Mathematics recommends that students in grades 3 through 5 should “develop a stronger understanding of the various meanings of multiplication and division, encounter a wide range of representations and problem situations that embody them, learn about the properties of these operations, and gradually develop fluency in solving multiplication and division problems” (NCTM 2000, p. 149).

### 1. Types of Multiplication and Division Problems

Think back to the childhood instruction you received in multiplication and division. The focus may have been on performing multidigit calculations accurately. Students’ ability to “do” multiplication and division calculations—to apply step-by-step procedures that result in correct solutions—implies that they have procedural knowledge of these operations. However, simply being able to perform calculations does not necessarily mean that students understand these operations. Conceptual knowledge is based on understanding relationships—in this case, relationships that represent multiplication and division. These relationships can be expressed using pictures, graphs, objects, symbols, and words. For example, when students encounter a word problem and are asked to translate the relationship into symbols, their conceptual knowledge of the situation helps them do so. Since everyday mathematics is almost always applied in the context of words, not symbols, it is important for students to understand the relationships inherent to multiplication and division problems.

Likewise, the language of multiplication and division situations must be understood. In a multiplication equation such as  $3 \times 2 = 6$ , the 3 and the 2 are *factors*. The 6, which is the solution to this equation, is the *product*. In division we sometimes

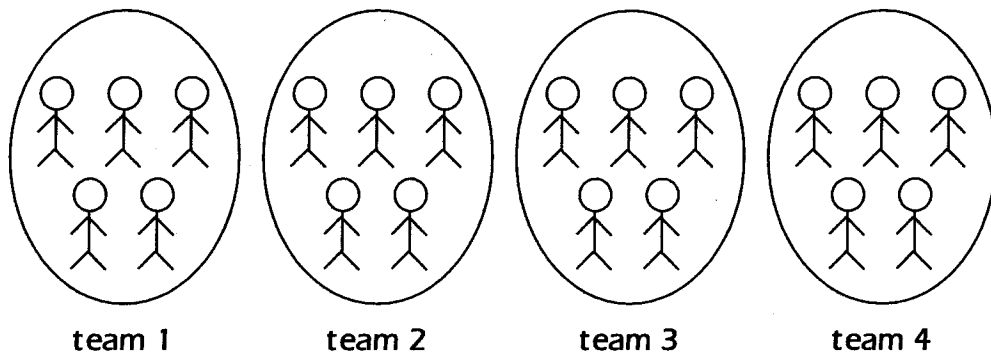
convert the operation to a multiplication equation ( $2 \times \square = 6$ , if we are dividing 6 by 2) and refer to the unknown as the missing factor, but more often we use separate labels. In the equation  $6 \div 2 = 3$ , for example, the 6 is the *dividend*, the 2 is the *divisor*, and the 3 (the answer) is the *quotient*. In some division expressions ( $7 \div 2$ , for example), there is also a *remainder*, which can be expressed in a number of different ways: as a whole number, a decimal, or a fraction. In the example given, the quotient can be represented as 3 remainder 1, 3.5, or  $3\frac{1}{2}$ . The relationship between these terms can be expressed generically this way: *factor*  $\times$  *factor* = *product* and *dividend*  $\div$  *divisor* = *quotient* + *remainder*.

Researchers have defined and classified multiplication and division problems in a number of different ways based on their semantic structure—that is, how the relationships are expressed in words. The semantic structure of problems differs with regard to the nature of the quantities used and the quantity that serves as the unknown. The semantic structure of multiplication and division problems has two broad categories: asymmetrical and symmetrical problems. In asymmetrical situations, the quantities play different roles. For example, in the problem *There are 15 cars in the parking lot and each car has 4 tires. How many tires are there in all?*, 4 represents the amount in one group, 15 represents the number of groups and also acts as the multiplier. These roles are not interchangeable. If you switch the numbers (4 cars with 15 tires each) you have a different problem: the number of things in one group is 15 and the number of groups is now 4. (In an asymmetrical problem the answer is the same regardless of the role of the quantities, but that is not obvious to children.) In symmetrical situations, on the other hand, the quantities have interchangeable roles. In symmetrical multiplication problems it is not clear which factor is the multiplier. For example, in the problem *What is the area of a room that is 10 feet by 12 feet?*, either number can be the width or the length and either can be used as the multiplier.

There are three subcategories of asymmetrical problems: (1) equal grouping, (2) rate, and (3) multiplicative compare. There are two subcategories of symmetrical problems: (1) rectangular array and (2) cross product. Finally, each subcategory includes both multiplication and division problems, depending on which quantity is unknown. Students need to work with all these types of multiplication and division problems in order to make sense of the relationships inherent to each type and to extend their understanding of these operations beyond mere procedures. As a teacher you need to guide students in discerning the role of the quantities within problems.

### *Equal Grouping Problems*

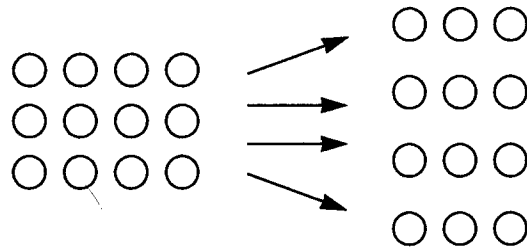
Usually when we think of the operation of multiplication, an equal grouping problem comes to mind. In equal grouping multiplication problems, one factor tells the number of things in a group and the other factor tells the number of equal-size groups. This second factor acts as a multiplier. For example, in the problem *There are four basketball teams at the tournament and each team has five players. How many players are at the tournament?*, the factor five indicates the number of players in one group and the factor four indicates the number of equal groups of five. In this case, the four acts as the multiplier. Equal grouping problems are easy to model with pictures or by using repeated addition:



Two situations result in a problem's being classified as an equal grouping division problem—either the number of groups is unknown or the number in each group is unknown. These two types of division situations are referred to as *quotitive division* and *partitive division*, respectively.

Here is a partitive division problem: *Twenty-four apples need to be placed into eight paper sacks. How many apples will you put in each sack if you want the same number in each sack?* The action involved in partitive division problems is one of dividing or partitioning a set into a predetermined number of groups. If students model this situation, 24 objects are evenly distributed into 8 different piles or groups.

When teaching division, teachers often choose partitive division examples to highlight equal sharing. For example, students are instructed to divide a set number of counters into four equal groups by distributing the counters one at a time into four piles:



Yet if partitive division problems are used exclusively in instruction, students often have difficulty making sense of quotitive division problems. Their mental model of what division is all about does not include this other meaning.

In quotitive division problems the number of objects in each group is known, but the number of groups is unknown. For example: *I have 24 apples. How many paper sacks will I be able to fill if I put 3 apples into each sack?* The action involved in quotitive division is one of subtracting out predetermined amounts. If asked to model this problem, students usually repeatedly subtract 3 objects from a group of 24 objects and then count the number of groups of 3 they removed (e.g., 8). (In partitive division, on the other hand, they “divide” the 24 objects into 3 groups.)



The standard long division algorithm uses the quotitive interpretation of division: the divisor represents the number in one group, and this amount is repeatedly subtracted from the dividend. The number of multiples (or groups) of the divisor that are subtracted from the dividend is the answer. If students only have had limited exposure to quotitive division examples, the action of subtracting rather than partitioning may not make sense to them; they may not “see” the link between division and repeated subtraction and simply perform the algorithm by rote.

### ***Rate Problems***

Rate problems involve a rate—a special type of ratio in which two different quantities or things are compared. Common rates are miles per gallon, wages per hour, and points per game. Rates are frequently expressed as unit rates—that is, one of the quantities in the ratio is given as a unit (e.g., price per single pound or miles per single hour). In rate problems, one number identifies the unit rate and the other tells the number of sets and acts as the multiplier. For instance, in the problem *Concert tickets cost seven dollars each. How much will it cost for a family of four to attend the concert?*, the unit rate is seven dollars per single ticket (seven to one) and the multiplier is four. Rate problems can also be expressed as division situations. Here is a partitive division rate problem (size of one group is unknown): *On the Hollingers’ trip to New York City, they drove 400 miles and used 12 gallons of gasoline. How many miles per gallon did they average?* Quotitive division problems (number of equal groups is unknown) are also common in this category: *Jasmine spent \$100 on some new CDs. Each CD cost \$20. How many did she buy?*

Since many students have had little experience with rates other than prices, these problems can be quite confusing. One way to help students understand rate multiplication problems is to have them calculate a unit rate (e.g., the number of feet they can walk or jog in one minute, the number of words they can read or write in one minute) and then apply this unit rate to various unit groupings (if I can read 45 words in one minute, I can read 90 words in two minutes, 135 words in three minutes, . . .) Especially difficult for young students are rates involving speed and distance. Not only are the ideas somewhat hard to grasp, but we refer to these two concepts in a number of different ways—by using a rate (miles per hour or miles per gallon) or by using words and phrases such as *speed*, *distance*, *how fast*, and *how far*.

### ***Multiplicative Compare Problems***

The third type of asymmetrical problem is multiplicative compare, also called a scalar problem. Here, one number identifies the quantity in one group or set while the other number is the comparison factor. For example, in the problem *Catherine read 12 books. Elizabeth read 4 times as many. How many books did Elizabeth read?*, the number 12 tells us the amount in a group and the number 4 tells us how many of these groups are needed. In multiplicative compare division problems, either the amount in each group or the comparison factor is missing. The following problems are multiplicative compare division problems:

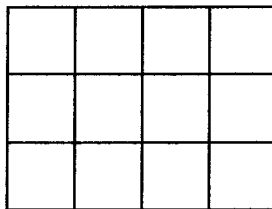
*Elizabeth read 48 books during summer vacation. This is 4 times as many as Catherine. How many books did Catherine read during summer vacation?*

*Elizabeth read 48 books during summer vacation. Catherine read 12 books during summer vacation. How many times greater is the number of books Elizabeth read compared with the number of books Catherine read?*

The relational language in multiplicative compare problems (e.g., *times as many as*, *times greater*) is difficult for all students, especially so for those for whom English is a second language. Students make sense of both the language and the relationships the language implies by discussing and modeling these problems. Comparing and contrasting the language of additive relationships (*more than*, *less than*) with that of multiplicative relationships (*times more than*, *times as many*) may also be helpful to students. Despite the confusing nature of these problems, the consensus among researchers is that early experience with multiplication and division should involve problems of this type.

### ***Rectangular Array Problems***

The rectangular array problem, a subcategory of symmetrical problem commonly known as an area problem, is often used to introduce the idea of multiplication. Students are presented with an array (e.g., three by four) and asked to label the two sides of the array and determine the total number of square units in the array:

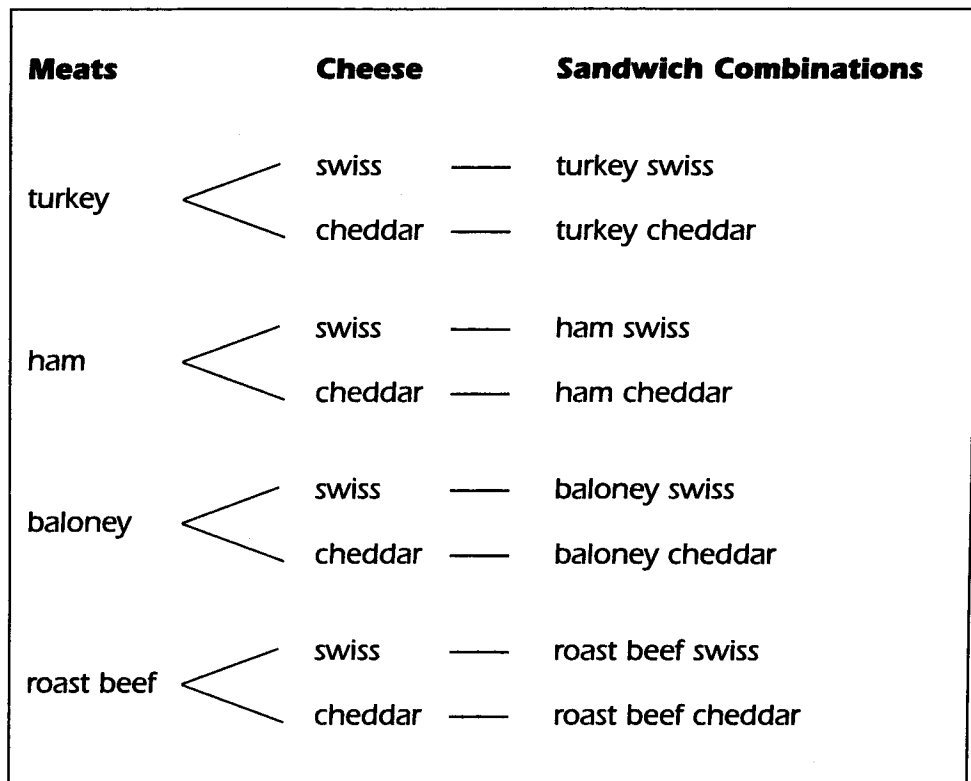


In rectangular array problems the role of the factors is interchangeable. For instance, when finding the area of the array above, neither the three nor the four is clearly the multiplier.

### ***Cross Product Problems***

The other subcategory of symmetrical problem consists of cross product or Cartesian product problems. Cross product problems entail a number of combinations. For example: *Pete's Deli stocks four types of cold cuts and two types of cheese. How many different sandwiches consisting of one type of meat and one type of cheese are possible?* In these problems, like the rectangular array problems, neither of the two factors is clearly the multiplier.

Students tend to use tree diagrams to solve cross product problems, in order to help them find all the combinations. For example, for the problem above, a student might use a diagram like this:

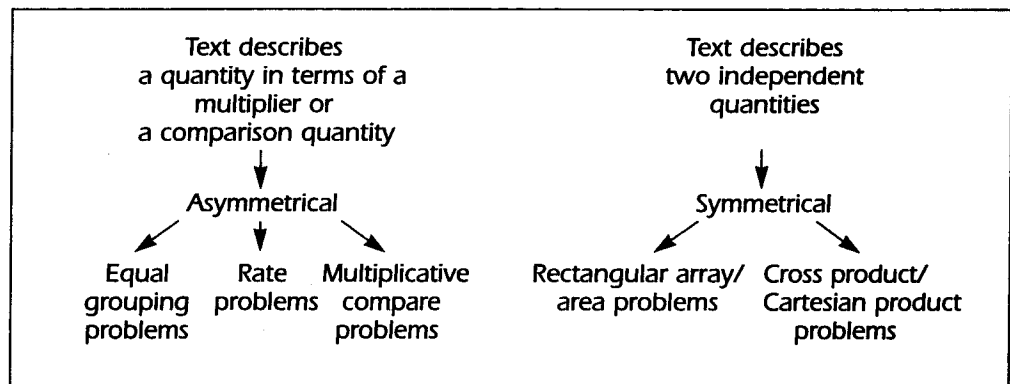


However, this diagram doesn't highlight how multiplication can be used to determine the number of combinations. Teachers often have to point out the relationship between the number of objects to be combined (4 and 2) and the final number of combinations ( $4 \times 2 = 8$ ).

Both rectangular array and cross product problems have related division problems called *missing factor division problems*. For example: *Susan has 24 different outfits consisting of a blouse and a pair of pants. She has 4 pairs of pants. How many blouses does she own?*

Although this problem can be solved by dividing 24 by 4, the relationship can also be expressed as  $4 \times \square = 24$ . Many students convert combination division problems to the inverse operation of multiplication, hence the name *missing factor*.

The diagram below shows how the different multiplication and division problems are related:



All the problem types should be used with students in order to help them extend their understanding of multiplication and division. Although it isn't essential to make the categories explicit for students or for them to learn the labels, categorizing problems can promote memory and understanding. Teachers should therefore be able to identify the different problem types and make them part of their instruction.

## Activity

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### Identifying and Classifying Problems

Determine whether the following problems are asymmetrical or symmetrical. Then decide on the category in which the problem belongs (equal grouping, rate, multiplicative compare, rectangular array, or cross product). Finally identify whether each problem is a multiplication or a division problem.

1. Five children are planning to share a bag of 53 pieces of bubble gum. How many will each get?
2. Pies cost \$7.50 each. Paula bought 5 pies. How much did they cost in all?
3. Peter bicycled 36 miles in 3 hours. How fast did Peter bike?
4. A restaurant offers 5 appetizers and 7 main courses. How many different meals can be ordered if a meal consists of one appetizer and one main course?
5. This year Mark saved \$420. Last year he saved \$60. How many times as much money did he save this year than last year?
6. This year Maddie saved 4 times as many dollars as she saved last year. Last year she saved \$18. How many dollars has she saved this year?
7. Each jar holds 8 ounces of liquid. If there are 46 ounces of water in a pitcher, how many jars are needed to hold the liquid?
8. Leroy has a 1,440-square-inch piece of fabric that is 60 inches wide. How long is the fabric?
9. Sam Slick is really excited about his new clothing purchases. He bought 4 pairs of pants and a number of jackets, and they all can be mixed and matched. Sam can wear a different outfit consisting of jacket and pants 12 days in a row. How many jackets did he buy?
10. Liana walked 12 miles at a rate of 4 miles per hour. How many hours did it take Liana to walk the 12 miles?
11. Maria earned \$24. She earned 4 times as much as Jill. How much did Jill earn?
12. The foundation of the house measures 70 feet by 25 feet. What is the square footage of the ground floor of the house?

### *Things to Think About*

How did you proceed in classifying problems? Many adults first sort the problems into two groups, either multiplication problems or division problems, and then identify the subcategories. Students usually have a difficult time sorting problems, since it requires a deep understanding of problem structure. This understanding develops gradually over time after extensive involvement with problems. How do you think students might react to problem 3? It is a rate problem, and the question asks how fast Peter biked. Many students do not realize that

miles per hour and speed (how fast) represent the same idea and are confused by this problem. Discussion is one way to address this discrepancy. ▲

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Are you surprised by the variety of multiplication and division problems that students have to be able to solve? Being able to make sense of and solve multiplication and division word problems is one aspect of understanding these operations. Research suggests that children in the elementary grades need experiences with a variety of asymmetrical and symmetrical problems. Otherwise, students' mental models of what constitutes multiplication and division may be seriously compromised. While it is not essential for students to identify a problem as asymmetrical or symmetrical, it may help them to be able to classify it as a multiplicative compare or an area problem. And it is important for students to consider the role of the quantities in these problems. Furthermore, in order to understand the relationships within these problems, students need to model situations and discuss the relationships between the quantities. They need to become comfortable determining different types of "unknowns." Traditional instruction rarely addresses the different features of problems and in some cases only includes examples from the equal grouping category. Instruction must include problems from all categories in order to challenge and expand students' existing mental models of multiplication and division.

## 2. Understanding Multiplication and Division

Multiplication and division are the visible part of an "enormous conceptual iceberg" (Vergnaud 1994, p. 46) that also includes ratio, rate, rational number, linear function, dimensional analysis, and vector space. A variety of phrases and words are used to describe multiplication and division situations. Knowledge that includes multiplication and division relationships is sometimes referred to as *multiplicative knowledge*. When we identify, understand, and use relationships between quantities that are based on multiplication or division, we are applying *multiplicative reasoning*.

Students' understanding of multiplication and division develops slowly—generally between grades 2 and 5. Understanding involves both conceptual and procedural knowledge—students must be able to identify multiplicative situations and be able to apply knowledge of multiplication and division facts and algorithms in finding a solution. How do children progress in their application of multiplicative reasoning?

In the early grades, students make sense of and solve simple multiplication and division problems without realizing that they are dealing with multiplicative situations. They tend to use additive solution strategies such as counting and repeated addition/subtraction. For example, to solve the problem *Karen bought four books at \$10.00 apiece. How much did they cost altogether?*, a student might add four 10s and represent the problem symbolically as  $10 + 10 + 10 + 10$  rather than as  $4 \times 10$ . Researchers such as Fischbein et al. (1985) suggest that additive reasoning,

which is based on joining and separating, is intuitive—it develops naturally through encounters with many situations in one’s environment (see chapter 3). Schooling builds on students’ intuitive understanding of addition and subtraction by focusing on additive situations in the early grades and then introducing situations that can be addressed through multiplicative reasoning in the later grades. In most cases, students require instruction to help them understand multiplicative relationships. Yet over time students develop an understanding of multiplicative situations and are able to determine whether to apply additive or multiplicative reasoning to solve specific problems (or to realize, as in the example above, that both can be used).

As students become better able to identify and make sense of multiplicative situations, they tend to start applying more sophisticated solution strategies. The progression of solution strategies from direct counting to repeated addition/subtraction to multiplicative operations is neither linear nor clear-cut. The strategies students employ depend on the size of the numbers in the problem, the semantic structure or subcategory of the problem, and the quantity that is unknown. Furthermore, the progression from additive strategies (counting and repeated addition/subtraction) to the use of multiplicative strategies requires several major cognitive shifts in students’ thinking.

To think multiplicatively, students must be able to consider numbers as units in a different way from when they are simply adding or subtracting. They must consider units of one and they must also consider units of more than one—composite units. For example, the number six in an additive situation can represent six discrete objects that can be counted and operated upon, yet in a multiplicative situation the six might represent a unit of six (or a group of six). In our place value system, multiplicative reasoning is applied to counting groups of 100, 10, and 1. The number 346 is equivalent to the sum of 3 composite groups of 100, 4 composite groups of 10 and 6 ones. Whether dealing with place value or operations, in order to reason multiplicatively, students must understand the idea that composite groups (groups of more than one) can be counted multiple times and operated on as an entity. They must also recognize that the quantitative relationship centers around equal-size groups. In order to help students make these cognitive shifts in reasoning, instruction in the early stages of learning multiplication and division should focus on helping students identify, create, and count composite, equal-size groups.

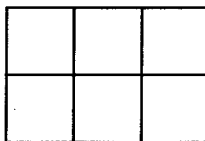
Another cognitive shift for students in progressing from additive to multiplicative reasoning is understanding the nature of the units of the quantities. In multiplication and division some of the quantities express relationships between two units (e.g., price per gallon, miles per hour, apples per bag). Students must make sense of these many-to-one relationships. Also, whereas in addition and subtraction students work with quantities of the same unit (e.g., eight dollars plus two dollars), in asymmetrical multiplication and division problems students work with quantities of different units (e.g., *Carla walked ten miles in two hours. What was her walking speed?*). Additive problems are “unit preserving” in that the solution quantity has the same label as the quantities in the problem. Multiplicative problems are “unit transforming” in that the units of the quantities in the problem are not the unit of the solution. For instance, considering the problem above, the units *miles* and *hours* are transformed into

a third unit, *miles per hour*. This is also true of symmetrical problems such as rectangular array problems: *Find the area of a room that is 8 feet by 12 feet*. In this case the two factors have the same unit, feet, but the solution, 96, has a completely different unit, square feet.

Other factors that influence children's understanding of multiplication and division are related to the models we use to teach these operations. The most common model is repeated addition (e.g.,  $5 \times 4$  is presented as  $4 + 4 + 4 + 4 + 4$ ) and usually involves problems categorized as equal grouping. This model is appropriate for introducing multiplication but is limiting if it is the only model used—repeated addition cannot be used to model multiplicative situations such as cross products and rates. Furthermore, while repeated addition builds on students' understanding of addition, if this is the only model students know, it suggests a very narrow view of the meaning of multiplication and division. In addition, the model of repeated addition/subtraction carries with it the restraint that the multiplier must always be a whole number, since repeatedly adding or subtracting portions of a group is difficult to conceptualize (e.g., interpreting  $\frac{1}{5} \times \frac{2}{3}$  as taking  $\frac{1}{5}$  groups of  $\frac{2}{3}$  and adding these groups together does not make a lot of sense and is counterintuitive). Compounding this difficulty, the emphasis on whole number multiplication and division problems leads elementary school students to generalize incorrectly that multiplication always results in products that are larger than the factors and division always results in quotients that are smaller than the dividend. Typically students will state, "Multiplication makes bigger, division makes smaller." Imagine how difficult it is for students to make sense of solutions in which both factors are fractions or decimals if they try to apply the repeated addition model to these problems.

This raises the question, what models should we use to help students learn about multiplicative situations? Repeated addition/subtraction definitely should be used in early instruction. However, it should not be the only model.

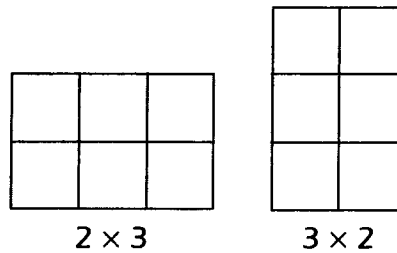
Rectangular arrays have proven to be useful because they can be linked to repeated addition. This array shows two rows of three columns:  $3 + 3 = 6$ :



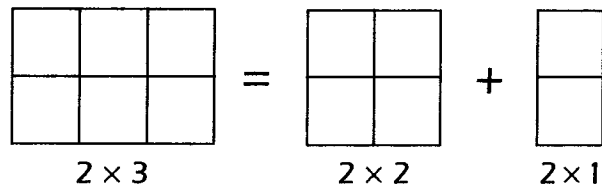
Cross product problems can also be shown using arrays. Six shirt-and-tie combinations can be made with two ties and three shirts:

	$S_1$	$S_2$	$S_3$
$T_1$	$T_1S_1$	$T_1S_2$	$T_1S_3$
$T_2$	$T_2S_1$	$T_2S_2$	$T_2S_3$

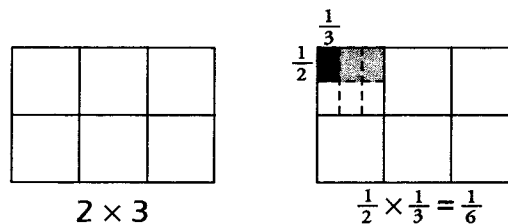
Furthermore, arrays can be used to illustrate the commutative and distributive properties of multiplication. The array below shows visually how  $2 \times 3$  is equivalent to  $3 \times 2$ :



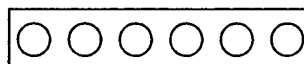
This array illustrates how  $2 \times 3 = (2 \times 2) + (2 \times 1)$ :



Finally, arrays can also be used to illustrate fraction and decimal multiplication. The one below can be used to help visualize  $\frac{1}{2} \times \frac{1}{3}$  ( $\frac{1}{3}$  of  $\frac{1}{2}$  of a square unit is shaded):

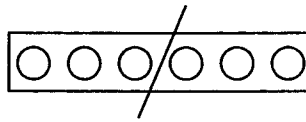


Another model teachers should consider using to illustrate multiplicative situations involves making a group or set and then acting upon the set in one of three ways: (1) taking a part of the set, (2) making several copies of the set, or (3) making several copies of the set and also taking a part of it. For example, start with a group or set of six objects:





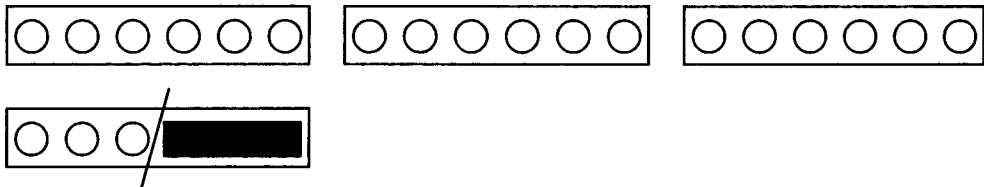
A part of the set ( $6 \times 0.5$ ) can be taken:



Several copies of the set can be made ( $6 \times 3$ ):



Finally, several copies can be made and a part taken ( $6 \times 3.5$ ):



Multiplicative compare problems can be illustrated using this model where the number in the starting set is one of the factors and the action on the set is determined by the comparison factor.

Cross product problems can be modeled in a variety of ways. First, as mentioned earlier, an array is a useful model. Matching strategies by listing the possible combinations is another model students often use. And tree diagrams help some students to consider the number of combinations.

<b>Ties</b>		<b>Shirts</b>		<b>Combinations</b>
Tie 1	/	Shirt 1	—	Tie 1, Shirt 1
	—	Shirt 2	—	Tie 1, Shirt 2
	\	Shirt 3	—	Tie 1, Shirt 3
Tie 2	/	Shirt 1	—	Tie 2, Shirt 1
	—	Shirt 2	—	Tie 2, Shirt 2
	\	Shirt 3	—	Tie 2, Shirt 3

Instructional techniques other than models can help students learn to reason multiplicatively by focusing on the numerical relationships between quantities. For example, teachers can introduce tables that show the proportional relationships in a rate problem, since it appears that students are better able to identify and understand the relationships between pairs of quantities when the information is presented this way. For the problem *Gumdrops cost \$3.89 per pound. How much will it cost to buy four pounds?*, the following table relates or links dollars and pounds:

	POUND		DOLLARS
	1	$\times \$3.89 \Rightarrow$	\$3.89
$(\times 4)$	4	$\times \$3.89 \Rightarrow$	$(\times 4)$ ?

The multiplicative relationship between pounds and dollars ( $\times 3.89$ ) as well as the multiplicative relationships between pounds and pounds ( $\times 4$ ) and dollars and dollars ( $\times 4$ ) is highlighted.

### 3. Factors, Multiples, and Divisors

There are many relationships within and between multiplication and division that need to be considered when teaching elementary school students. Important topics include prime factors (see page 16), multiples, and divisors. All of these topics are related: in order for a number to be divisible (i.e., with a remainder of zero) by another number, the divisor must be a factor of the dividend. Young students often talk informally of how a “number divides evenly with none left over” when expressing this idea of divisibility. The rules regarding divisibility are related to factors and divisors. (The most common divisibility rules are described in activity 2.) Divisibility rules or tests usually are introduced to students in grades 4, 5, or 6. Knowing about the divisibility rules contributes to one’s ability to make sense of numbers. For example, divisibility rules can be applied when determining factors and checking to see whether a number is a multiple of another number. These rules are very handy when considering options in problem situations (for example, can 57 band members be grouped by threes?), especially since the calculations can be done mentally.

#### Activity



#### Divisibility Rules

Review the rules regarding divisibility by 2, 3, 4, 5, 6, 8, 9, and 10. (There is also a divisibility test for 7, but it is neither simple nor efficient and thus is not included; checking to see if a number is divisible by 7 is best done with a calculator.) Develop a divisibility rule for 15 and for 18.

Divisibility by 2: The number is even.

Divisibility by 3: The sum of the digits is divisible by 3.

Divisibility by 4: Either the last two digits are 00 or they form a number that is divisible by 4.

Divisibility by 5: The ones digit of the number is either 0 or 5.

Divisibility by 6: The number is even *and* the sum of the digits is divisible by 3.

Divisibility by 8: Either the last three digits are 000 or they form a number that is divisible by 8.

Divisibility by 9: The sum of the digits is divisible by 9.

Divisibility by 10: The ones digit of the number is 0.

### **Things to Think About**

What is the number 234,567,012 divisible by? It is divisible by 2 because it is an even number. It is divisible by 3 because the sum of the digits is 30 ( $2 + 3 + 4 + 5 + 6 + 7 + 0 + 1 + 2 = 30$ ) and 30 is divisible by 3. It is divisible by 4 because the last two digits form the number 12, and 12 is divisible by 4. It is divisible by 6 because it is both an even number and divisible by 3. It is not divisible by 5, 8, 9 or 10.

The fact that 234,567,012 is not divisible by 8 when it is divisible by 2 and 4 leads us to consider how the divisibility of a number relates to the factors of that number. In order for a number ( $n$ ) to be divisible by another number ( $a$ ), it must possess all of the prime factors of that number. The prime factors of 8 are  $2 \times 2 \times 2$ . For 234,567,012 to be divisible by 8 it must have  $2^3$  in its prime factorization: 234,567,012 has  $2 \times 2$  in its prime factorization because it is divisible by 4 and even, but it does not have the extra 2 as a prime factor to make it divisible by 8.

To determine whether a number is divisible by 15, consider the prime factors of 15—3 and 5. Applying divisibility rules for both 3 and 5 will determine whether a number is divisible by 15. The same method can be used for 18: apply the divisibility rules for 2 and for 9 to any number to determine whether it is divisible by 18. Why can't you use the divisibility rules for 3 and 6 to determine whether a number is divisible by 18? Again, it is related to factors. The prime factors of 18 are  $2 \times 3 \times 3$ . Thus, for a number to be divisible by 18, it must possess *all* of those factors. When you use the divisibility rule for 6, you are checking to see whether a number contains the factors 2 and 3. When you use the divisibility rule for 3, you are again checking to see whether the number has 3 as a prime factor (it does—you determined that with the 6 rule), but you aren't checking to see whether the number has two 3s ( $3 \times 3$ ) as factors.

Another example may help clarify this idea: using the divisibility rules, 24 is divisible by 3 because  $2 + 4 = 6$  and 6 is divisible by 3; 24 is divisible by 6 because it is even and is divisible by 3; but 24 is not divisible by 18 even though it is divisible by 3 and 6; why not? The rule for divisibility by 3 and the rule for divisibility by 6 both check to see whether 3 is a factor of 24 (it is), but the rules do not check to see whether 9 ( $3 \times 3$ ) is a factor of 24 (it isn't). ▲

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What about division problems that have a remainder? What are the relationships between the remainder and the other numbers in a division problem? What patterns do remainders form? (Patterns when remainders are written as decimals are investigated on page 105 in chapter 6.)



## Recurring Remainders

Divide each number on the chart below by 5. Circle all numbers that when divided by 5 have a remainder of 1. Draw a square around the numbers that when divided have a remainder of 2. Place a triangle around those numbers that when divided by 5 have a remainder of 3. Underline the numbers that have a remainder of 4. Describe the patterns. How do the patterns change if the divisor is 7 instead of 5?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

### Things to Think About

What patterns occurred? Numbers with a remainder of 1 when the divisor is 5 all end in 1 or 6 and are in the first and sixth columns. This is because numbers with a 0 or 5 in the one's place are divisible by 5, and if the number has a remainder of 1 it must be one more than a multiple of 5. The numbers that have remainders of 2, 3, or 4 fall into the subsequent columns because they are each 2, 3, or 4 more than a multiple of 5.

Why can't you have a remainder of 5 or greater when you divide by 5? If the remainder is 5 or more, another group of 5 can be removed from the dividend. Students often make the mistake of having too big a remainder when learning long division, especially when their understanding of the long division algorithm is limited:

$$\begin{array}{r} 7 \text{ r } 7 \\ 5 \overline{)42} \\ \underline{35} \\ 7 \end{array}$$

$$\begin{array}{r} 8 \text{ r } 2 \\ 5 \overline{)42} \\ \underline{40} \\ 2 \end{array}$$

What do you notice about the remainders you get when you divide the numbers on the chart by a different divisor? How do the patterns change? How do they stay the same? The number of remainders is always one less than the divisor ( $n - 1$  when  $n$  is the divisor). Whereas the patterns of remainders when the divisor is 5 fall into columns, other visual remainder patterns form diagonals. The patterns are all based on the multiples of the number. For example, since the multiples of 7 are 0, 7, 14, 21, 28, 35, 42, 49, . . . , the remainder patterns are 1, 2, 3, 4, 5, and 6 more than these multiples. Thus the numbers 1, 8, 15, 22, 29, 36, 43, and 50 all have remainders of 1 when divided by 7. These numbers fall along diagonals. ▲

What happens to the remainder in division situations? Is the remainder always used? Is the quotient ever rounded up or down because of the remainder? Can the remainder be the solution to a problem?

## Activity



### Interpreting Remainders

Examine each of the following problem situations and decide what happens to the remainder. What factors affected your decisions?

1. You have a rope that is 25 feet long. How many 8-foot jump ropes can you make?
2. You have 30 toys to share fairly with 7 children. How many will each child receive?
3. The ferry can hold 8 cars. How many trips will it have to make to carry 42 cars across the river?
4. Six children are planning to share a bag of 50 large cookies. About how many will each get?
5. You have a 10-foot wooden board that you want to cut into 4 pieces. How long will each piece be?
6. Kinne picked 14 quarts of blueberries to make jam. Each batch of jam uses 3 quarts of berries. How many quarts of blueberries will Kinne have left for muffins?

### *Things to Think About*

There are many different ways of interpreting the remainder in a division problem. It depends primarily on the context of the problem. The first problem illustrates a situation in which the remainder is not used. If the jump ropes are to be exactly eight feet long, then three jump ropes can be made and the extra one foot of rope will be tossed or saved for another project. The second problem is similar; each child will receive four toys. Perhaps the two extra toys will be given away or passed along between the seven children at periodic intervals. At any rate, cutting the extra toys into fractional parts makes no sense. These two problems illustrate quotitive and partitive division, respectively. A remainder can occur whether the number of groups or the number within one group is being determined.

In problem 3, the quotient, 5.25, must be rounded up. Five trips of the ferry will not get all of the cars across the river and a portion of a trip is impossible. Thus, six ferry trips are required. A great deal of research has been conducted on students' interpretation of problem situations in which the quotient must be rounded up to the next whole number because of the remainder. Do students really believe that an answer such as 5.25 trips is reasonable and possible? Investigation has revealed that students understand you cannot have a part of certain situations such as ferry crossings but that they are so accustomed to doing computations without a context they simply don't bother to check whether their answer makes sense. The implication for us as teachers is that we must focus more instruction on interpretation of answers (perhaps using estimation) so that students apply common sense in determining what to do with remainders. (For example, students might be asked to suggest the kind of answer that would make sense in the situation in problem 3 rather than to find the solution.)

Sometimes a remainder is expressed as a fraction or as a decimal. Problems 4 and 5 illustrate contexts in which this makes sense. Since the cookies in problem 4 are large, it might be possible to divide the last two cookies into thirds so that each child receives  $8\frac{1}{3}$  cookies. However, if a bag of hard candies were being divided, the remainder would be dealt with differently. In problem 5 the plank of wood is divided into four pieces, each  $2\frac{1}{2}$  feet long. In the case of measurements such as lengths and weights, it usually makes sense to record the remainder as a fraction or as a decimal.

In the last problem, division is used to find the solution but the remainder is the answer. Situations such as these take students by surprise simply because they don't expect the remainder to be the solution. This example again points to the need to provide students with experiences interpreting different types of remainders. As teachers we must help students understand how the context of a problem affects the interpretation of remainders and lead discussions about the various roles of the remainder based on the types of numbers and labels. ▲

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Another topic related to multiplication and division involves multiples. Multiples are first introduced in school when children learn to skip count. Later they are used when multiplying and when finding least common denominators in order to work with fractions. Eventually students will generalize relationships with factors of numbers to find multiples of algebraic expressions.

## Activity



### Patterns With Multiples

List the first twenty-one multiples of 3. What patterns do you notice? Is 423 a multiple of 3?

#### *Things to Think About*

Multiplying 3 by consecutive whole numbers (e.g., 0, 1, 2, 3, . . . ) gives the set of multiples of 3. (Zero is a multiple of all numbers since any number times zero equals zero.) The first twenty-one multiples of 3 are 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, and 60. Some of the patterns you might have observed in the first twenty-one multiples are:

- ▲ The multiples alternate between even and odd numbers.
- ▲ There are groupings of three or four multiples for each tens digit (12, 15, 18 or 30, 33, 36, 39).
- ▲ The units digit repeats after a cycle of 10 (0, 3, 6, 9; 2, 5, 8; 1, 4, 7).
- ▲ Since multiples of 3 are divisible by 3, the sum of the digits of each multiple is also a multiple of 3.

When students are asked to list the multiples of a number and are encouraged to look for patterns, they will find the ones mentioned and more! Looking for patterns builds familiarity with multiples of particular numbers, helps students link skip counting and multiplication, and highlights the orderliness of numerical relationships. It also may help some students remember some of the multiplication

facts. Furthermore, the patterns can be used to explore relationships between operations such as repeated addition and multiplication.

Why do the multiples alternate between even and odd numbers? The odd number 3 is being added to obtain each subsequent multiple ( $0 + 3 = 3$ ,  $3 + 3 = 6$ ,  $6 + 3 = 9$ ,  $9 + 3 = 12$ , . . .). An even number plus an odd number sums to an odd number, and an odd number plus an odd number sums to an even number (see page 5 in chapter 1).

Why are the groupings of multiples in sets of three or four? The connection between divisors and multiples produces this pattern. If you consider the first set of ten numbers (0–9), three groups of 3 can be created. Consider the next two sets of ten (10–19 and 20–29). Three groups of 3 can be made from each set, each with a remainder of 1. With the fourth set of ten (30–39) not only can three groups of 3 be made, but the extra remainder can be matched with the earlier remainders to form another group of 3 (30, 33, 36, 39). Four groups of 3 will occur again in the 60s (60, 63, 66, 69) when the extra remainders from earlier tens are combined.

Why are multiples of 3 divisible by 3? It is related to why the divisibility rule for 3 works. Consider any three-digit number:  $abc$ . It can be expressed as  $(a \times 100) + (b \times 10) + (c \times 1)$  or  $a(99 + 1) + b(9 + 1) + c$ . This can be rewritten as  $99a + a + 9b + b + c$  or  $(99a + 9b) + (a + b + c)$ . The number  $(99a + 9b)$  is divisible by 3 because 99 and 9 are divisible by 3. Thus, if  $(a + b + c)$  is also divisible by 3 then the original three-digit number ( $abc$ ) is divisible by 3 and so are all subsequent multiples of 3. Applying this to a number such as 423, we have:

$$\begin{aligned}423 &= (4 \times 100) + (2 \times 10) + (3 \times 1) \\ &= 4(99 + 1) + 2(9 + 1) + 3 \\ &= [99(4) + 4] + [9(2) + 2] + 3 \\ &= [99(4) + 9(2)] + (4 + 2 + 3)\end{aligned}$$

Examine  $[99(4) + 9(2)] + (4 + 2 + 3)$ . The first part of the expression— $[99(4) + 9(2)]$ —is divisible by 3 because  $99(4)$  is divisible by 3 and  $9(2)$  is divisible by 3. Therefore, to check whether the original number, 423, is divisible by 3, add the digits— $(4 + 2 + 3)$ . Since  $4 + 2 + 3 = 9$  and 9 is divisible by 3, then 423 is also divisible by and a multiple of 3. ▲

## Activity



### Least Common Multiples

As you solve the following somewhat more perplexing problems, consider how the concept of least common multiple is used in the solution process.

1. You may have noticed that hot dog rolls always come in packages of eight, whereas hot dogs are packaged in groups of ten. What are the fewest packages of buns and hot dogs you have to buy so that you have the same number of hot dogs and buns?
2. One lighthouse takes 45 minutes to make a complete revolution, and a second light house revolves every 30 minutes. If the lighthouses are synchronized to begin a revolution at 9:00 A.M., how long will it take before they are again synchronized?

3. The life cycle of a species of cicadas is 17 years. The last time this particular species emerged during a presidential election year was 1996. (Presidential elections occur every four years.) When will this species next emerge during a presidential election year?

### **Things to Think About**

In the first problem did you simply multiply 8 and 10 to get 80—a common multiple of both 8 and 10? Yet you don't need 80 buns and hot dogs. Here's why: 8 and 10 have a common factor of 2:  $8 = 2 \times 4$  and  $10 = 2 \times 5$ . The least common multiple has to have all of the factors of 8 ( $2 \times 4$ ) and all of the factors of 10 (the 2 is already accounted for but the 5 is not), but no other factors (or you have common multiples). So the factors needed are  $2 \times 4 \times 5$ , for a product of 40. The least common multiple is 40; four packages of hot dogs ( $4 \times 10$ ) and five packages of hot dog rolls ( $5 \times 8$ ) will give the same number of hot dogs and buns.

Problem 2 is similar to the first one. A common multiple must be found in order to determine when the lighthouses again will be synchronized. Because 45 and 30 have some common factors ( $45 = 15 \times 3$  and  $30 = 15 \times 2$ ), common multiples less than 1,350 ( $45 \times 30$ ) exist. Yet the least common multiple must have all of the factors of both numbers. These factors are  $15 \times 3 \times 2$ . Thus, the least common multiple is 90—in 90 minutes, or at 10:30 A.M., the two lighthouses will again be synchronized.

In the final problem the two numbers, 17 and 4, have no common factors. In this case, the least common multiple is found by multiplying 17 and 4 to obtain a product of 68. This species of cicadas will again emerge during a presidential election year in 2064 ( $1996 + 68$ ). ▲

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We can use what we know about remainders, divisibility, and multiples to solve a wide range of number theory problems. For example, what number when divided by both 5 and 7 has a remainder of 2? If the number has a remainder of 2 when divided by 5 and by 7 it has to be two more than a multiple of 5 and 7. You can use the chart in activity 5, mark the numbers with remainders of 2, and then look for the intersection, or you can list numbers that are two more than the multiples.

Numbers divided by 5 with remainder 2: 7, 12, 17, 22, 27, 32, 37, 42, 47 . . .

Numbers divided by 7 with remainder 2: 9, 16, 23, 30, 37, 44, 51, 58, 65 . . .

The least common multiple of 5 and 7 is 35. Two more than 35 is 37, so 37 is one solution to this problem. The next number that when divided by 5 and 7 has a remainder of 2, is 72 since the next multiple of 5 and 7 is 70 and two more is 72.

## **Teaching Multiplication and Division**

Students need to develop a strong and complete conceptual knowledge of multiplication and division. If they don't, they will have difficulty with more advanced multiplicative situations such as proportions, measurement conversion, linear functions, and exponential growth. Therefore, they need to explore many different types of asymmetric and symmetric word problems. Instruction should build on students'



intuitive understanding of additive situations by using the repeated addition models but must also include other models and strategies in order to extend students' understanding of these operations. Whole numbers, decimals, fractions, and integers should be used with all problem types in order to dispel the common misconception that "multiplication makes larger, division makes smaller." Instruction must also include opportunities for students to explore the role of factors, different types of units, and properties of multiplication and division.