Session 6

Number Theory

Key Terms in This Session

Previously Introduced
- counting numbers
- factor
- factor tree
- prime number

New in This Session
- composite number
- greatest common factor
- least common multiple
- Venn diagram

Introduction

As part of our exploration of number theory, we will look at two models for finding least common multiples and greatest common factors: the Venn diagram model and the area model. Later in the session, we will explore prime and composite numbers.

For information on required or optional material, see **Note 1.**

Learning Objectives

In this session, you will do the following:

- Understand greatest common factors and least common multiples and how they relate to one another
- Understand alternative models and methods for computing greatest common factors and least common multiples
- Understand prime and composite numbers
- Understand the location of prime numbers within the number system, and use this understanding to determine whether very large numbers are prime

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Note 1.

Materials Needed:
- Graph paper for the hands-on activity

Part A: Models for Multiples and Factors (75 min.)

The Venn Diagram Model

The numbers 24 and 36 have certain things in common, including many common factors—numbers that divide evenly into both of them. For example 2, 3, and 6 are all common factors. The largest such number is called the "greatest common factor." In this case, the greatest common factor of 24 and 36 is 12. No number greater than 12 is a factor of each of these numbers. **[See Note 2]**

Another characteristic numbers can share is a common multiple—a third number that is evenly divisible by both 24 and 36. The smallest such number is called the "least common multiple." In this case, the least common multiple is 72. No number less than 72 is evenly divisible by each number.

One way to explore the common factors and multiples of the two numbers is to use a Venn diagram:
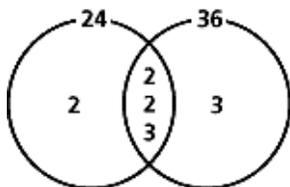
The circle on the left contains all the prime factors (i.e., counting numbers that have exactly two factors: themselves and 1) of 24, and the circle on the right contains all the prime factors of 36. (The number 1 doesn't qualify as prime, because it has only one factor.)

The numbers contained in the intersection are those factors that are in both numbers; i.e., their common factors. That means that the 2s and the 3 in the intersection, both separately and multiplied together (2 • 2, 2 • 3, and 2 • 2 • 3—or 2, 3, 4, 6, and 12), are all common factors.

Common Factors of 24 and 36
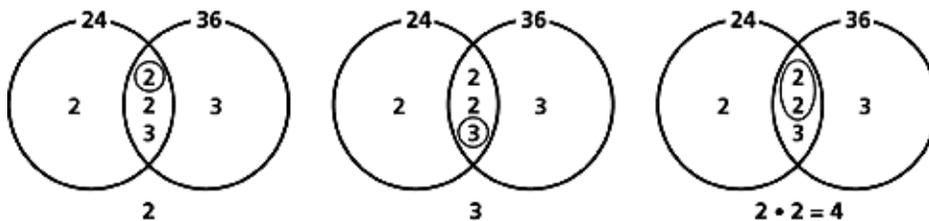
Greatest Common Factor
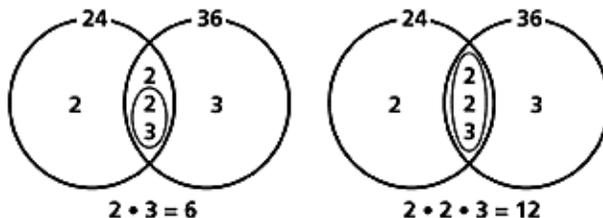
Note that the largest of these factors is 12. The greatest common factor (GCF) of 24 and 36 is 2 • 2 • 3, or 12, the product of all the numbers in the overlap.

Since the circle on the left contains all the factors of 24, every multiple of 24 must contain all of these factors. Likewise, since the circle on the right contains all the factors of 36, every multiple of 36 must contain all of these factors.

Note 2. The greatest common factor is equivalent to the greatest common divisor. The greatest counting number that evenly divides a and b is both the greatest common factor and the greatest common divisor of both a and b.

Part A, cont'd.

In order to be a multiple of both numbers, a number must contain all the factors of both numbers. The smallest number to do this is 2 • 2 • 2 • 3 • 3, or 72, the product of all the factors in the circles. Thus, the least common multiple (LCM) of 24 and 36 is 72.

Least Common Multiple
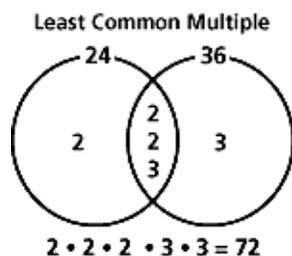
[See Note 3]

Problem A1. Use a Venn diagram to determine the GCF and LCM for 18 and 30.

Finding Prime Factors

The numbers we've examined so far have been fairly simple to factor. Now let's look at a general method for finding prime factors. **[See Note 4]**

One method is to draw a factor tree. To do this, write a number—24, for example—and then draw an upside-down V under it. This V represents two "branches" of the factor tree. Think of a pair of numbers with the product 24; for example, 4 and 6. Check to see if either of these numbers is prime. In this case, the answer is no.

Draw another V under each number that is not prime, and find two factors for each of these numbers. In this case, we will find the factors 2 and 2 for 4, and 2 and 3 for 6. Now we have four factors, 2, 2, 2, and 3, all of which are prime numbers. This is the prime factorization of 24.

Note 3. The LCM and GCF can be difficult concepts to understand because we hear the words in the opposite order of their importance: For example, for the LCM, first we hear "least," then we hear "common," and last we hear "multiple." However, the most important word of the three is "multiple." The multiples of 24 are 1 • 24, 2 • 24, 3 • 24, and so forth, and the multiples of 36 are 1 • 36, 2 • 36, 3 • 36, and so forth. The next most important word is "common." We are looking for a number that is common to both numbers. The third most important word is the one we hear first, "least."

So the number we want is a multiple, common to both numbers, and the least of all such numbers. This would be the same case for the GCF, for which we want a number that is a factor, common to both numbers, and the greatest of all such numbers. It would be worth taking the time to have a class discussion of the three words when introducing the LCM and GCF.

Note 4. When you're asked to find factors, pay careful attention to the specific question that is posed to you. Were you told to find the prime factors, the prime factorization, the number of factors, or all the factors? These are four very different questions.

For example, for the number 36, the following statements are all true:

- The prime factors are 2 and 3.
- The prime factorization is $2^2 • 3^2$.
- The number of factors is nine.
- Those nine factors are 1, 2, 3, 4, 6, 9, 12, 18, and 36—1 • 36, 2 • 18, 3 • 12, 4 • 9, and 6 • 6. (Note that we only count the 6 once. This is why squares have an odd number of factors.)

Part A, cont'd.

In mathematics, we like to do things via consistent algorithms. So rather than just picking two factors, it's a good idea to make the process more consistent by first finding the smallest prime factor and its partner and then repeating that process on the partner (since the first number is guaranteed to be prime). For example, you could factor out all the possible 2s, then all the 3s, then 5s, 7s, and so on, until the number is completely factored. Here is what the diagrams look like for the numbers 24 and 36:
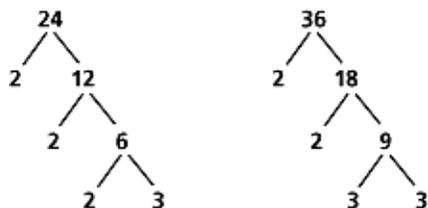
Problem A2. Does the order in which you factor a number matter? Is the product uniquely that one number? To answer these questions, use a factor tree to find the prime factorization of 60 in the following ways:

    a. Start by factoring out 2s.

    b. Do another diagram, but this time start by factoring out 10s.

    c. Do a third diagram, but this time start by factoring out 6s.

    d. What is the same and what is different about your results?

When you factor a number, no matter where you start, you always get the same set of factors; the only difference might be the order in which they occur. This phenomenon is called the fundamental theorem of arithmetic, which states that any integer (other than 0, and ±1) can be factored into a product of prime numbers and that this product is unique except for the order of the factors. This is another reason why 1 cannot be considered prime—otherwise, this, and every other result that builds on it, falls apart. For example, we could factor 6 in an infinite number of ways:

$$6 = 2 \cdot 3$$
$$= 1 \cdot 2 \cdot 3$$
$$= 1^2 \cdot 2 \cdot 3$$
$$\dots$$
$$= 1^{100} \cdot 2 \cdot 3$$

... and so on, for any number of 1s that we cared to use.

Problem A3. Draw a factor tree to find the factors of 231 and 195.

Problem A4. Use a Venn diagram to find the GCF and LCM of 231 and 195.

Part A, cont'd.

The Area Model

The area model makes the process of finding GCFs and LCMs visual. **[See Note 5]**

Greatest Common Factor

If we think of the numbers 24 and 36 as the dimensions of a rectangle, then it follows that any common factor could be the dimensions of a square that would tile that entire rectangle.

For example, a 1-by-1 square would tile the 24-by-36 rectangle without any gaps or overlaps. So would a 2-by-2 or a 3-by-3 square. Notice that these numbers are all common factors of 24 and 36.

To determine the GCF, we want to find the dimensions of the largest square that could tile the entire rectangle without gaps or overlap. Here's one quick method.
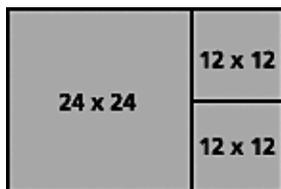
Start with the 36-by-24 rectangle:
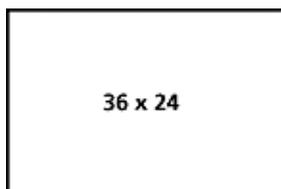
The largest square tile that fits inside this rectangle and is flush against one side is 24 by 24. Only one tile of this size will fit:
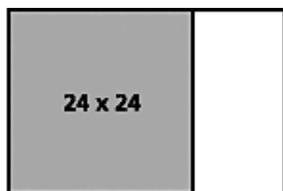
The largest square tile that fits inside the remaining rectangle and is flush against one side is 12 by 12. Two tiles of this size will fit. The original rectangle is now completely filled:

Note that the 24-by-24 square could also be filled with the 12-by-12 tiles, so 12 by 12 is the largest tile that could fill the original 24-by-36 rectangle; therefore, 12 is the GCF of 24 and 36.

Note 5. The area model shows how to fill a rectangle with squares (to find the GCF) or make a square with rectangles (to find the LCM). It can be a useful method for visual learners.

Part A, cont'd.

Least Common Multiple

Conversely, if we think of 24 and 36 as the dimensions of a rectangle that could tile a square, then it follows that any common multiple could be the dimensions of a square that could be tiled by this rectangle.

For example, since 24 • 36 = 864, a square that is 864 by 864 could be tiled by the 24-by-36 rectangle. The LCM of 24 and 36 would be the dimensions of the smallest square that could be tiled by the 24-by-36 rectangle. Here's a quick method for determining the LCM.

Start with the 24-by-36 rectangle. Your goal is to make a square tiled with rectangles of these dimensions:
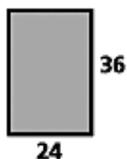
Since the width (24) is less than the height (36), add a column of tiles to the right of the rectangle (in this case, one tile). This makes a 48-by-36 rectangle:
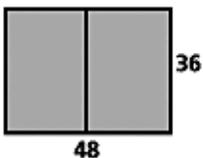
The width (48) is now greater than the height (36), so add a row of tiles under the existing rectangle (in this case, two tiles). This makes a 48-by-72 rectangle:
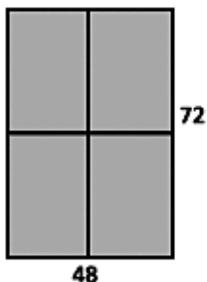
The width (48) is now less than the height (72), so add another column (two tiles) to the right of the existing rectangles. The dimensions are now 72 by 72—and you've made a square!
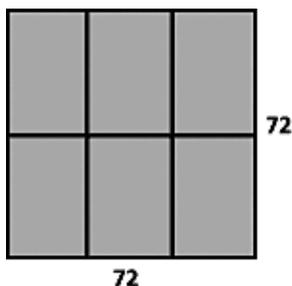
The 72-by-72 square is the smallest square that can be tiled with a 24-by-36 rectangle. Therefore, the LCM of 24 and 36 is 72.

Use the area models as outlined above to answer Problem A5. Use graph paper when drawing the squares and rectangles you wish to represent to ensure that the dimensions of the shapes are drawn to scale.

Part A, cont'd.

Problem A5. Use the area model to find the GCF and LCM of the following:
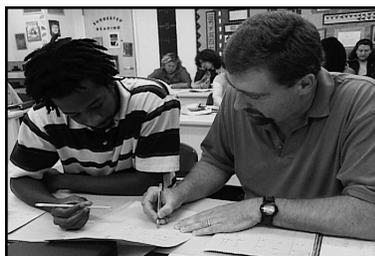
    a. 30 and 42

    b. 18 and 30

Video Segment (approximate time: 6:04-8:05): You can find this segment on the session video approximately 6 minutes and 4 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Ben and Doug use the area model to find the GCF for two numbers, following the analogy of tiling a rectangle with the biggest square they can fit. Watch this segment after you've completed Problem A5.

Notice that the teachers omitted one step and didn't use the square with the dimensions of 12 by 12 to tile the 12-by-30 rectangle. Think about why going through all the steps will ensure that the result will be the largest common factor rather than just any common factor.

Write and Reflect

Problem A6. Can you explain in your own words why the area model works?

Video Segment (approximate time: 8:32-10:09): You can find this segment on the session video approximately 8 minutes and 32 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Here, Ben and Doug use the area model to find the LCM for two numbers, following the analogy of finding the biggest square that can be tiled with a rectangle whose dimensions are the two original numbers. Notice the connection they make between the area model and Venn diagrams.

Part B: Looking for Prime Numbers (45 min.)

Locating Prime Numbers

In this part, we'll continue a mathematics tradition begun by Eratosthenes of Cyrene (276-194 B.C.E.)—the same person who's known for accurately estimating the diameter of the Earth based on shadows cast from the Sun's light.

Eratosthenes worked out a method, now called the "Sieve of Eratosthenes," to collect all the prime numbers and allow all composites (multiples of prime numbers) to "drain through." He used a grid that looked like what we now call the 100 board—the first row is 1-10, the second row 11-20, etc. This grid does locate the prime numbers, but it does not help us understand where to look for them. If you try looking for prime numbers in this grid, you will discover that it's not so easy to locate them in a systematic way:

```
 1  2  3  4  5  6  7  8  9  10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
 :  :  :  :  :  :  :  :  :  :
 :  :  :  :  :  :  :  :  :  :
```

In the following activity, we will use a different grid to locate the prime numbers. This grid has only six columns, starting with the numbers 2 through 7. As you will see, such positioning of numbers will make the patterns more noticeable and consequently will be more helpful in answering the question of where the prime numbers are located.

```
 2   3   4   5   6    7
 8   9  10  11  12   13
14  15  16  17  18   19
20  21  22  23  24   25
26  27  28  29  30   31
32  33  34  35  36   37
38  39  40  41  42   43
44  45  46  47  48   49
 :   :   :   :   :    :
 :   :   :   :   :    :
```

To work on the following problems, you may want to use the copy of the above grid on page 127.

Problem B1. Circle the 2, which is a prime number. Next, cross out all the multiples of 2, as they are not prime numbers.

   a. Imagine that the grid goes on forever. Present a convincing argument for the fact that all the numbers in the first, third, and fifth columns are multiples of 2 and would thus be crossed out.

   b. Could multiples of 2 be located in any other column?

Part B, cont'd.

Problem B2. Next, circle the smallest remaining prime number (i.e., 3). Cross out all the multiples of 3, as they are not prime.

    a. Again imagine that the grid goes on forever. Present a convincing argument for the fact that all the numbers in the second and fifth columns are multiples of 3 and would thus be crossed out. (Of course, the fifth column is already gone because it contains multiples of 2!)

    b. Could multiples of 3 be located in any other column?

Problem B3. Again, circle the smallest remaining prime number (i.e., 5) and cross out all the multiples of 5, as they are not prime. Notice that these multiples are not all located in particular columns, so crossing them out is not as easy as before. Continue in a similar manner until all the numbers on the grid are either circled or crossed out.

Examine the grid and imagine that it extends to infinity. Where on the extended grid should you look to find prime numbers greater than 3?

Video Segment ((approximate time: 10:56-15:22): You can find this segment on the session video approximately 10 minutes and 56 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch how Professor Findell and the participants worked on Problems B1-B3 in this video segment.

Are you convinced that the conjecture about locating prime numbers is true? Try extending the table by a few more rows to see if the rule still applies.

Necessary and Sufficient Conditions

The previous activities illustrate the difference between a necessary and a sufficient condition. We have shown that every prime greater than 3 is located in either the fourth or sixth columns of our grid. This means that it is *necessary* for the number to be located in one of those two columns if it is prime and greater than 3.

However, we also found numbers in those columns that are composites (not prime); thus, that location is not a *sufficient* condition for a number to be prime. This type of thinking is very useful when analyzing relationships in mathematics.

To give another example: For a number to be divisible by 6, it is necessary, but not sufficient, that it is divisible by 3. Conversely, for a number to be divisible by 3, it is sufficient, but not necessary, that it is divisible by 6.

It is *necessary* for the rest of the primes to fall in either the fourth or sixth columns, but that is not a *sufficient* condition.

Problem B4.

    a. Will thinking about what you know about the location of prime numbers help you check whether 943,787,589 is prime?

    b. How about whether 532,391 is prime?

Part B, cont'd.

Is This Number Prime?

Finding factors and checking if a large number is prime remains one of the most time-consuming tasks in mathematics. Even very powerful computers cannot do this task quickly. For this reason, prime numbers are very useful in cryptography. Secret messages are sometimes coded using large numbers that are the product of two large prime numbers.

Here are the rules you can use to find out whether a particular number is prime:

- Pick a number n.

- Start with the least prime number, 2. See if 2 is a factor of your number. If it is, your number is not prime.

- If 2 is not a factor, check to see if the next prime, 3, is a factor. If it is, your number is not prime.

- Keep trying the next prime number until you reach one that is a factor (in which case n is not prime), or you reach a prime number that is equal to or greater than $\sqrt{n}$.

- If you have not found a factor less than or equal to $\sqrt{n}$, you can be sure that your number is prime.

Let's try a number; for example, 97. To check if 97 is a prime number, we start a list of factors, as above:

1 · 97

2 · (no number works here)

3 · (no number works here)

(4 is not prime; it doesn't need to be checked, because we know 2 didn't work)

5 · (no number works here)

(6 is not prime; it doesn't need to be checked, because 2 and 3 didn't work)

7 · (no number works here)

(8 is not prime; it doesn't need to be checked, because 2 and 4 didn't work)

(9 is not prime; it doesn't need to be checked, because 3 didn't work)

This brings us to 10. Ten is greater than the square root of 97, and therefore its partner on the right would have to be less than 10. Since we've already checked every number less than 10, we know that none of them are factors of 97. Therefore, 97 is prime.

Problem B5.

    a. Why don't you need to check any prime numbers greater than 11 to see if 127 is prime?

    b. Can you explain the rule for where to "stop"?

Problem B6.

    a. Use this method to determine if 257 is prime.

    b. What is the greatest prime you need to check?

Problem B7. Use this method to determine if 359 is prime.

Part B, cont'd.

Video Segment (approximate time: 16:50-18:24): You can find this segment on the session video approximately 16 minutes and 50 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Liz and Jeanne try to determine whether a large number is prime. They apply divisibility rules and computational shortcuts to solve the problem. Watch this segment after you've solved Problems B5-B7.

Were the methods you came up with similar or different?

In Part A of this session, we looked for prime factors of numbers. Using this method, the prime factorization of 72 is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$, or $2^3 \cdot 3^2$. In Part B, we listed all the factors of the number 72 in ascending order: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72.

Notice that these two lists are very different. The prime factorization, which the fundamental theorem of arithmetic says is unique, lists only the prime factors of 72 and lists each factor as many times as it appears.

In contrast, the list of all factors lists all numbers that are factors of 72, many of which are not prime. When doing problems, make sure to think about this distinction.

Homework

Problem H1. Prime numbers have exactly two factors. Now find some numbers that have exactly three factors. What do these numbers have in common? That is, how would you categorize these numbers? **[See Tip H1, page 128]**

Problem H2. There is a way to find the number of factors of a positive integer without writing out all the factors, and it requires finding the prime factorization first. This problem will help you discover that rule.

Go through the table, and list all the factors for each number. Then in the table enter the total number of factors (including the number itself and 1). Look for patterns, and try to write a general rule for the number of factors for any integer.

Integer	Prime Factorization	Number of Factors
2	2^1	2
4	2^2	
8	2^3	
16	2^4	
2^n	2^n	
3	3^1	2
9	3^2	
27	3^3	
81	3^4	
3^m	3^m	
6	$2^1 \cdot 3^1$	4
12	$2^2 \cdot 3^1$	
18	$2^1 \cdot 3^2$	
36	$2^2 \cdot 3^2$	
$2^n \cdot 3^m$	$2^n \cdot 3^m$	

Problem H3. A number is called a perfect number if the sum of all of its factors is equal to twice the value of the number. What are the two smallest perfect numbers?

Problem H4. An abundant number is one in which the sum of its factors is greater than twice the number. A deficient number is one in which the sum of its factors is less than twice the number. Which numbers less than 25 are abundant and which are deficient?

Problem H5. You have seen that every prime number greater than 3 is one less or one more than a multiple of 6. It is also true that every prime number greater than 2 is one more or one less than a multiple of 4. How would you prove this fact?

Prime Number Grid

2   3   4   5   6   7

8   9   10   11   12   13

14   15   16   17   18   19

20   21   22   23   24   25

26   27   28   29   30   31

32   33   34   35   36   37

38   39   40   41   42   43

44   45   46   47   48   49

$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$

Tips

Homework

Tip H1. Look for numbers with three factors, not three prime factors. The number itself and 1 are always factors, so there must be exactly one other factor. When we factor a number, we typically get two distinct factors. How could we get only one new factor?

Solutions

Part A: Models for Multiples and Factors

Problem A1. 18 = 2 • 3 • 3, and 30 = 2 • 3 • 5. The factors they have in common are 2 and 3. In the left circle is 3, and in the right circle is 5:
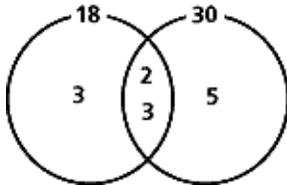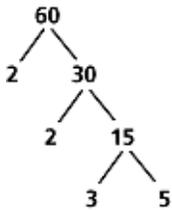
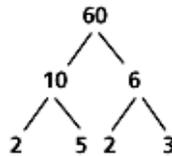The GCF is the product of all the numbers in the intersection: 2 • 3 = 6. The GCF is 6.

The LCM is the product of all the numbers in the Venn diagram: 3 • 2 • 3 • 5 = 90. The LCM is 90.
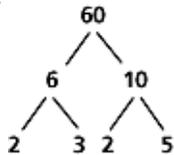
Problem A2.

a.

b.

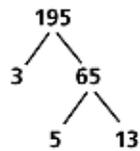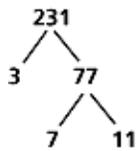c.

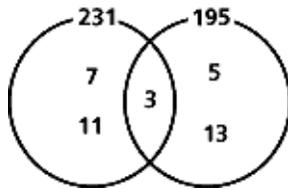d. All three methods yield the same prime numbers (two factors of 2, one factor of 3, and one factor of 5).

Problem A3.
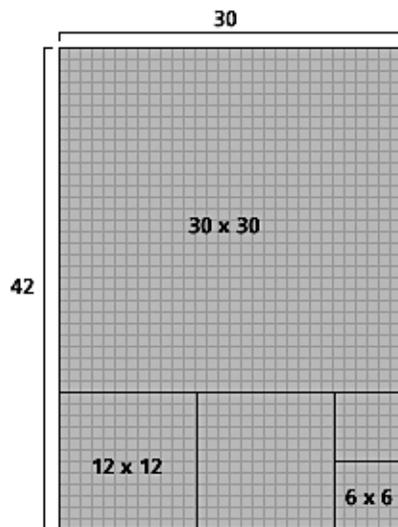
Solutions, cont'd.

Problem A4.

The intersection contains the only common factor, which is 3. On the left are the factors of 231 that are not factors of 195; i.e., 7 and 11. On the right are the factors of 195 that are not factors of 231; i.e., 5 and 13.

The GCF is the product of all the numbers in the intersection (in this case, just the 3). The GCF is 3.

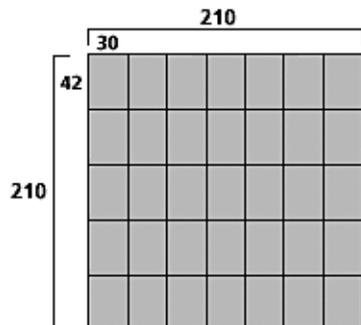The LCM is the product of all the numbers in the Venn diagram: $7 \cdot 11 \cdot 3 \cdot 5 \cdot 13 = 15{,}015$. The LCM is 15,015.

Problem A5.

a. The largest square that can tile this entire rectangle without any gaps or overlap is 6 by 6. Therefore, the GCF of 30 and 42 is 6.

The smallest square that could be tiled by this rectangle is 210 by 210. Therefore, the LCM of 30 and 42 is 210.
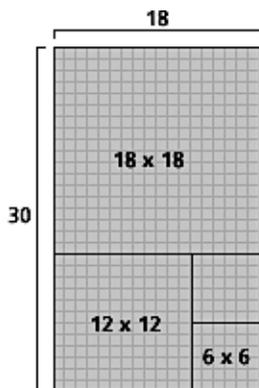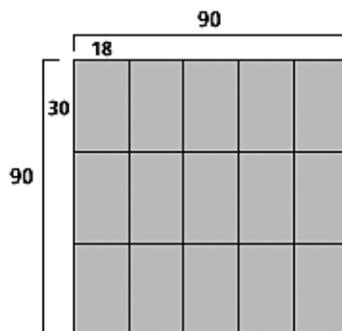
Solutions, cont'd.

Problem A5, cont'd.

b. The largest square that can tile this entire rectangle without any gaps or overlap is 6 by 6. Therefore, the GCF of 18 and 30 is 6.

The smallest square that could be tiled by this rectangle is 90 by 90. Therefore, the LCM of 18 and 30 is 90.

Problem A6. Answers will vary.

Part B: Looking for Prime Numbers

Problem B1.

a. All the numbers in the first, third, and fifth columns end in 2, 4, 6, 8, or 0, so they are all multiples of 2. (Refer to Session 5 if you need a refresher.)

Another convincing argument is that each number in a column is six more than the number directly above it. In other words, to move down a column, you add 6. Since 2 divides 6, 2 will divide any even number plus 6. So, as you move down a column, you will continue to get multiples of 2.

Yet another convincing argument is that because there are six numbers in each row, and 2 is a factor of 6, then each row has the multiples of 2 in the same position.

b. No. The number after an even number (or multiple of 2) is always odd, so any number in a column directly after an even number cannot also be even. As in the previous answer, all numbers in the second, fourth, and sixth columns are multiples of 6 more than the top number in the column. So the numbers in the second, fourth, and sixth columns are all a multiple of 6 more than an odd number and are therefore odd numbers themselves.

Problem B2.

a. All the numbers in the second and fifth columns have digits that add to a multiple of 3. Thus, referring back to divisibility tests, we know that all those numbers are divisible by 3.

Another argument is that because 3 is a factor of 6, each row has the multiples of 3 in the same position. Numbers in the second column are all a multiple of 6 more than 3 and are thus multiples of 3. Numbers in the fifth column are all multiples of 6.

b. No. In six consecutive numbers, there cannot be more than two multiples of 3. Numbers that are a multiple of 6 more than 2, 4, 5, or 7 will not be multiples of 3. In other words, if you take a column where every number is a multiple of 3, then every number in the column before that is one less than a multiple of 3— thus, it cannot be a multiple of 3. Likewise, every number in the column after it is one more than a multiple of 3 and thus cannot be a multiple of 3.

Solutions, cont'd.

Problem B3. The only numbers that can be prime numbers greater than 3 are the numbers that have not been crossed out yet. All of the remaining prime numbers must be located in the fourth or sixth columns, because the other columns are all crossed out.

In particular, the first number we have not crossed out yet must be prime. After crossing out multiples of 2 and 3, the first number we have not crossed out yet is 5.

The only two columns that have numbers not already crossed out are on either side of the column that contains all multiples of 6. That means that any prime number greater than 3 has to be either one more or one less than a multiple of 6.

Problem B4.

a. It may, if we can determine what column that number is in. That is, if the number is not one more or one less than a multiple of 6, then it is not prime. To do this, we can check divisibility by 2 and 3. The number is not divisible by 2 (its units digit is not even). To check if it is divisible by 3, add the digits: 9 + 4 + 3 + 7 + 8 + 7 + 5 + 8 + 9 = 60, which is a multiple of 3. So this nine-digit number will be in one of the crossed-out columns and thus is not prime.

b. The number 532,391 is not divisible by 2 or 3, and therefore is not a multiple of 6. To further check its location, you need to check the divisibility by 2 of the numbers one more and one less than 532,391; both clearly are. Then you need to check divisibility by 3 of the numbers one more and one less. A quick check shows that the digits of 532,392 sum to 24, so it's divisible by 3. This means that the number 532,392 is divisible by 6 and thus is located in the fifth column. The number 532,391 is located in the fourth column, and it may or may not be a prime.

Problem B5.

a. We only need to check prime numbers up to the square root of the given number. The square root of 127 is between 11 (the square root of 121) and 12 (144), so we only need to check the prime numbers up to 11.

b. By definition of square roots, n will factor as $\sqrt{n} \cdot \sqrt{n}$. Now think of finding other factors from this "middle point." If you change the first number to be something larger than $\sqrt{n}$, the second factor must get smaller to make the product stay constant at n. So you're guaranteed that when you factor n into a product of exactly two numbers, at least one of the two will be less than or equal to $\sqrt{n}$. Since this is true of any pair of two factors, it's certainly true if we restrict one of the factors to be a prime as well.

Problem B6.

a. The number 257 is not divisible by 2, 3, 5, 7, 11, or 13. It is prime.

b. The greatest prime number we need to check is 13, since it is the largest prime number less than the square root of 257 (which is just over 16).

Problem B7. We must check all prime numbers up to 17 (since the square root of 359 is just under 19). Since 359 is not divisible by 2, 3, 5, 7, 11, 13, or 17, it is prime.

Solutions, cont'd.

Homework

Problem H1. The number must be a square; otherwise, it would have an even number of factors. Try some square numbers:

1: 1 (one factor)

4: 1, 2, 4 (three factors)

9: 1, 3, 9 (three factors)

16: 1, 2, 4, 8, 16 (five factors)

25: 1, 5, 25 (three factors)

36: 1, 2, 3, 4, 6, 9, 12, 18, 36 (nine factors)

49: 1, 7, 49 (three factors)

The first four numbers with this property are 4, 9, 25, and 49. The next three after that are 121, 169, and 289. In general, the way to categorize these numbers is that they are squares of the prime numbers.

Problem H2. Here is the completed table:

Integer	Prime Factorization	Number of Factors
2	2^1	2
4	2^2	3
8	2^3	4
16	2^4	5
2^n	2^n	$(n + 1)$
3	3^1	2
9	3^2	3
27	3^3	4
81	3^4	5
3^m	3^m	$(m + 1)$
6	$2^1 \cdot 3^1$	4 (i.e., 2 · 2)
12	$2^2 \cdot 3^1$	6 (i.e., 3 · 2)
18	$2^1 \cdot 3^2$	6 (i.e., 2 · 3)
36	$2^2 \cdot 3^2$	9 (i.e., 3 · 3)
$2^n \cdot 3^m$	$2^n \cdot 3^m$	$(n + 1) \cdot (m + 1)$

To find the number of factors for any number, write the prime factorization of your number. Then record one more than each exponent. The number of factors will be the product of the augmented exponents.

So, for $72 = 2^3 \cdot 3^2$, the exponents are 3 and 2. One more than each exponent gives the numbers 4 and 3. The number of factors is 4 · 3, or 12.

Solutions, cont'd.

Problem H3. For this problem and the next, we need the following list of numbers and the sum of their factors:

Number	Factors	Sum
1	1	1
2	1, 2	3
3	1, 3	4
4	1, 2, 4	7
5	1, 5	6
6	1, 2, 3, 6	12
7	1, 7	8
8	1, 2, 4, 8	15
9	1, 3, 9	13
10	1, 2, 5, 10	18
11	1, 11	12
12	1, 2, 3, 4, 6, 12	28
13	1, 13	14
14	1, 2, 7, 14	24
15	1, 3, 5, 15	24
16	1, 2, 4, 8, 16	31
17	1, 17	18
18	1, 2, 3, 6, 9, 18	39
19	1, 19	20
20	1, 2, 4, 5, 10, 20	42
21	1, 3, 7, 21	32
22	1, 2, 11, 22	36
23	1, 23	24
24	1, 2, 3, 4, 6, 8, 12, 24	60
25	1, 5, 25	31
26	1, 2, 13, 26	42
27	1, 3, 9, 27	40
28	1, 2, 4, 7, 14, 28	56

The first two perfect numbers are 6 and 28, since their factors add to exactly twice the value of the number.

Problem H4. Refer to the table for Problem H3.

Abundant numbers less than 25 are 12, 18, 20, and 24. All others (besides 6, which is perfect) are deficient.

Problem H5. You could use a sieve-like table similar to the one used in this session:

You can see that the first and the third columns get crossed out immediately. Thus, the prime numbers will be located in the second or fourth columns.

Alternatively, you could argue that every number is either zero, one, two, or three more than a multiple of 4. If a number is zero or two more, then it can't be prime (unless it's 2), for such numbers are divisible by 2. This leaves "one more" and "three more" as the only choices. Three more than a multiple of 4 is the same as one less than the next multiple of 4. So again, the prime numbers will be located in the columns that are one more or one less than a multiple of 4.