MATHematics illuminated

FACILITATOR GUIDE

UNIT 6

UNIT 06

THE BEAUTY OF SYMMETRY

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(30 minutes)

Facilitator's note: This activity covers basic ideas that can be skipped with a more-experienced audience in favor of activities two and three.

This activity explores in more depth some of the ideas from Unit 6 having to do with sets, groups, integers, and doing math with things other than numbers.

A

1. Under which operations is the set of positive integers closed? How about the positive odd integers? The positive even integers? Give justification for your conclusions.
Answer: The positive integers are closed under addition and multiplication but not under subtraction or division. The positive odd integers are closed under multiplication but not under any of the other basic four operations.
The positive even integers are closed under addition and multiplication but not under subtraction or division. Justifications will vary, but should include examples: 5 – 6 = –1 shows that the set of positive integers is not closed under subtraction; 6 ÷ 5 shows that it is not closed under division; odd + odd = even shows that the positive odds are not closed under addition; odd × odd = odd shows that the odds are closed under multiplication; even + even = even shows that the evens are closed under addition; and even × even = even shows that they are closed under multiplication.

B

Let S = {a, b, c, d}.
Let x = an arbitrary member of S

* symbolizes an operation that follows the following rules:

a * x = a
x * a = a
b * x = x
x * b = x
b * b = b
c * d = d * c = d
c * c = d * d = c

ACTIVITY 1

1. Create the operation table for * on S.

Answer:

*	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	d
d	a	d	d	c

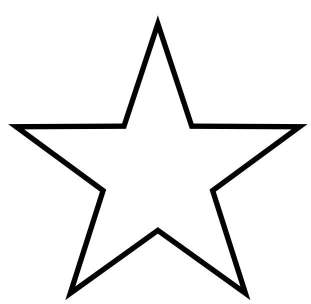
2. Is S a group under *? Explain.

Answer: No—no element of S serves as an Identity under *, so S cannot be a group under *.

C

1. Show that any combination of rotational symmetries of a regular five-pointed star (labeled I for the Identity rotation and $R_{\#\ \text{of degrees}}$ for each of the four other rotations) gives one of the basic rotational symmetries.

Answer: A full circle of rotation has 360 degrees; consider the top point to be 0 and perform all rotations clockwise. There are five points, so any rotation of 72 degrees (or a multiple of this) will bring the points into alignment with the original orientation. With the top point designated 0 degrees, the five points of the circle correspond to the measurements 0, 72, 144, 216, and 288 degrees.



ACTIVITY 1

2. Create a table showing this.

Answer:

*	I	R_{72}	R_{144}	R_{216}	R_{288}
I	I	R_{72}	R_{144}	R_{216}	R_{288}
R_{72}	R_{72}	R_{144}	R_{216}	R_{288}	I
R_{144}	R_{144}	R_{216}	R_{288}	I	R_{72}
R_{216}	R_{216}	R_{288}	I	R_{72}	R_{144}
R_{288}	R_{288}	I	R_{72}	R_{144}	R_{216}

D

Let's say we have a mystery rotation *x*. Use the symmetry table that you found in exercise C to solve the following equations:

Let * denote "followed by."

1. $R_{72} * x = R_{144}$

Answer: $x = R_{72}$

2. $R_{288} * x = R_{144}$

Answer: $x = R_{216}$

3. $(x * R_{144}) * R_{288} = R_{216}$

Answer: $x = R_{144}$

4. The system:

$(x * R_{216}) * y = R_{144}$

$(R_{72} * x) * y = I$

Hint: It may be easier just to reason this out than to try to apply standard algebraic techniques, because we haven't defined the inverse operation of *.

Answer: x = R72 and y = R216

There are other solutions in which x * y = R144.

ACTIVITY 2

(50 minutes)

Facilitator's note 1: This goes with Activity 3.

Facilitator's note 2: It would be good to forestall the following potential error: some people compute motions with respect to the object they are rotating rather than the fixed coordinate axes. This is fine for rotations, but it messes up reflections and leads to a structure that is not a group.

In this activity you will explore the symmetries of various regular polygons. The symmetries of regular polygons form what are collectively called the "dihedral" groups. "Dihedral" literally means "two-sided" in the sense that a coin has two sides. In order to explore and catalog the symmetries of these shapes, you will need to be aware of both sides.

To aid in your exploration, there are marked polygons for you to cut out, assemble, and use. Each polygon has a front and a back; you should cut out both and either glue or tape them together so that the labeled vertices match up exactly.

For each shape you should identify all of its symmetries. In order to have consistent notation, you can call the rotational symmetries R_0, R_1, ...R_n and the reflection symmetries S_0, S_1...S_n. For convenience, the axes of reflection have been labeled on each shape. In order to stay consistent, all rotations should be clockwise, and let R_0 represent the identity, (a rotation of 0 degrees).

For each shape you should fill in a "Cayley" table—that is, a table that shows how the different symmetries combine to form other symmetries, sort of like a multiplication table. The table structures are already constructed for you; you just need to fill them in. One thing to keep in mind is that if you reflect a shape, then the direction of rotation is reversed (imagine watching the reflected image of a clock).

Facilitator's note: it may help to demonstrate how to rotate or reflect shapes if participants are having trouble. For a review of how to combine symmetries to create other symmetries, please see the text for Unit 6.

After you have completed the tables, there are some questions that will lead you to understand the group structure of an arbitrary regular polygon...an "n-gon."

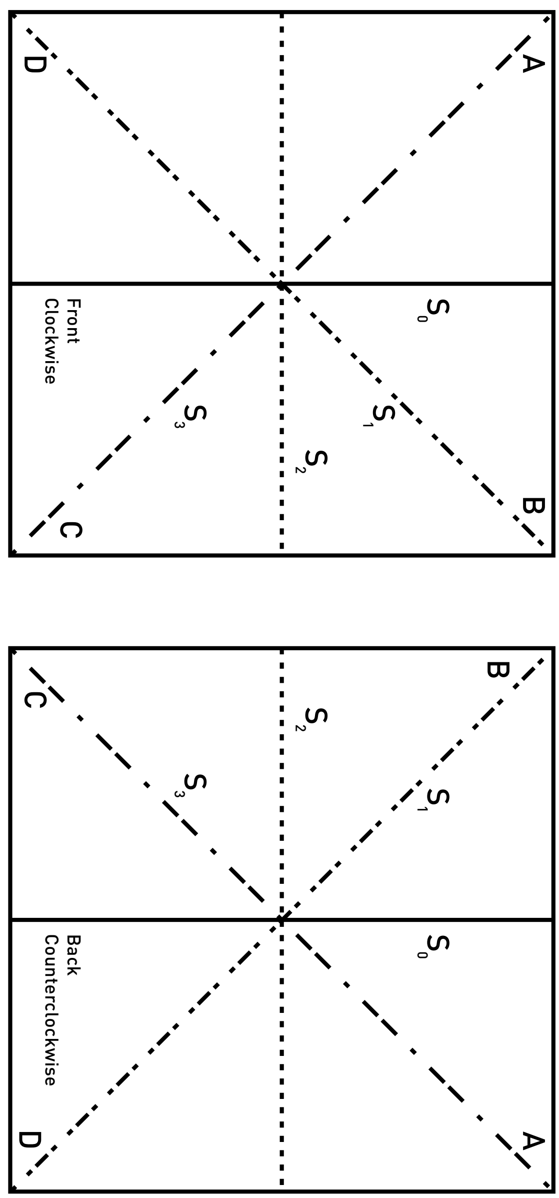
Please work as a group and feel free to divide the labor as you see fit. Enjoy!

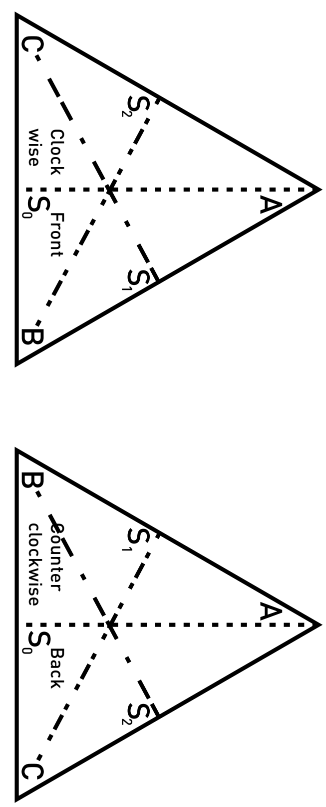
ACTIVITY 2

Worksheet 1 templates:

EQUILATERAL TRIANGLE AND SQUARE

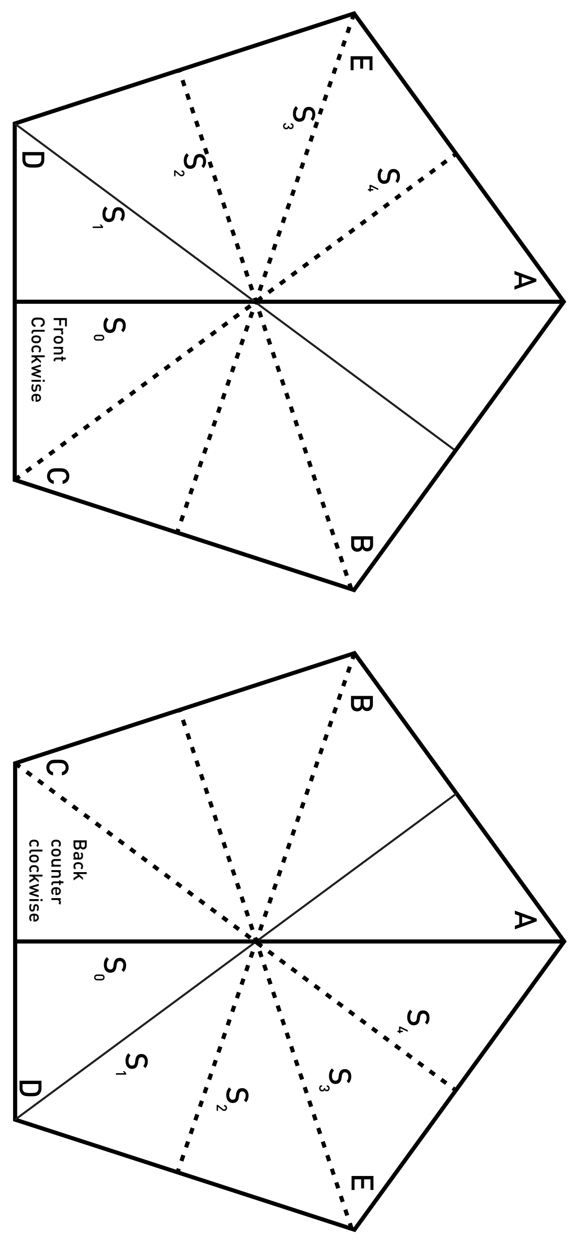
Regular triangle, square and pentagon with rotations and refelctions marked to construct Cayley Tables.





ACTIVITY 2 Worksheet 2 templates:

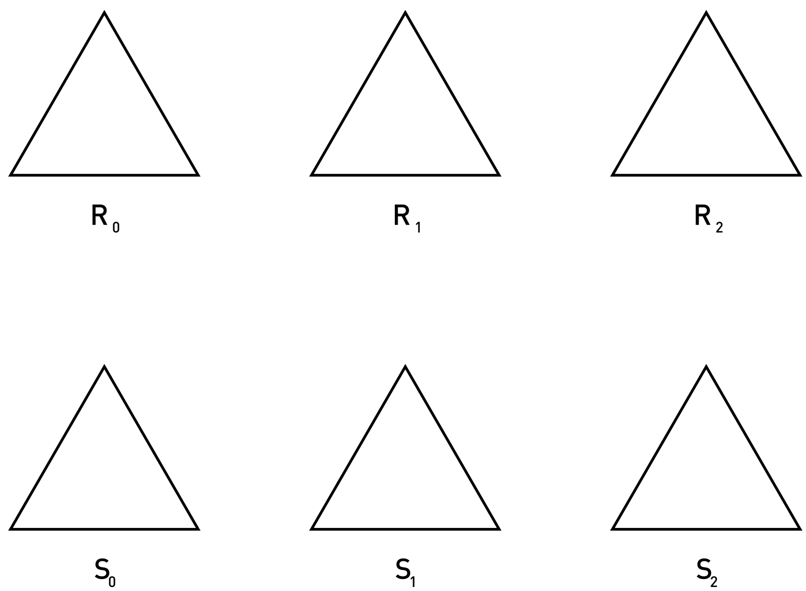
REGULAR PENTAGON



ACTIVITY 2

Worksheet 3 Equilateral triangle:

SYMMETRY TABLE



CAYLEY TABLE FOR EQUILATERAL TRIANGLE

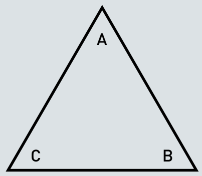
*	R_0	R_1	R_2	S_0	S_1	S_2
R_0						
R_1						
R_2						
S_0						
S_1						
S_2						

ACTIVITY 2

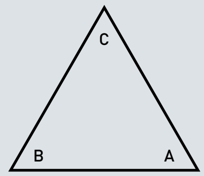
SYMMETRY TABLE

Worksheet 3 Equilateral triangle:

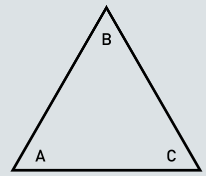
Answers:



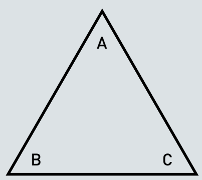
R_0



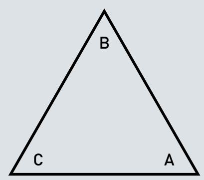
R_1



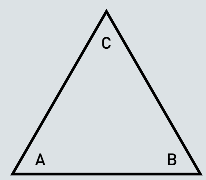
R_2



S_0



S_1



S_2

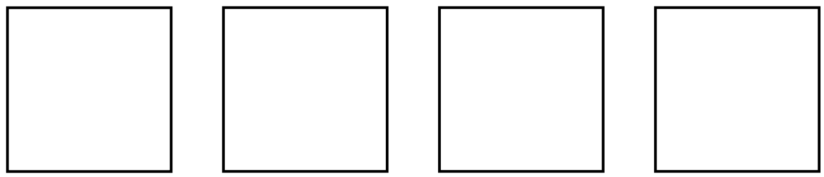
CAYLEY TABLE FOR EQUILATERAL TRIANGLE

*	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0	R_1	R_2	S_0	S_1	S_2
R_1	R_1	R_2	R_0	S_1	S_2	S_0
R_2	R_2	R_0	R_1	S_2	S_0	S_1
S_0	S_0	S_2	S_1	R_0	R_2	R_1
S_1	S_1	S_0	S_2	R_1	R_0	R_2
S_2	S_2	S_1	S_0	R_2	R_1	R_0

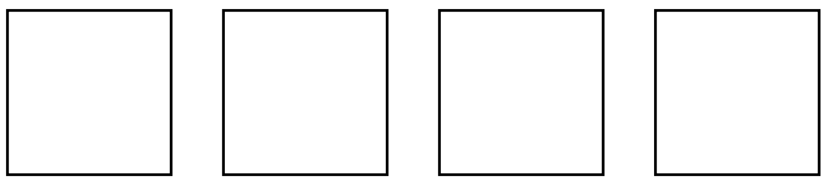
ACTIVITY 2

Worksheet 4 Square:

SYMMETRY TABLE



R_0 R_1 R_2 R_3



S_0 S_1 S_2 S_3

CAYLEY TABLE FOR EQUILATERAL TRIANGLE

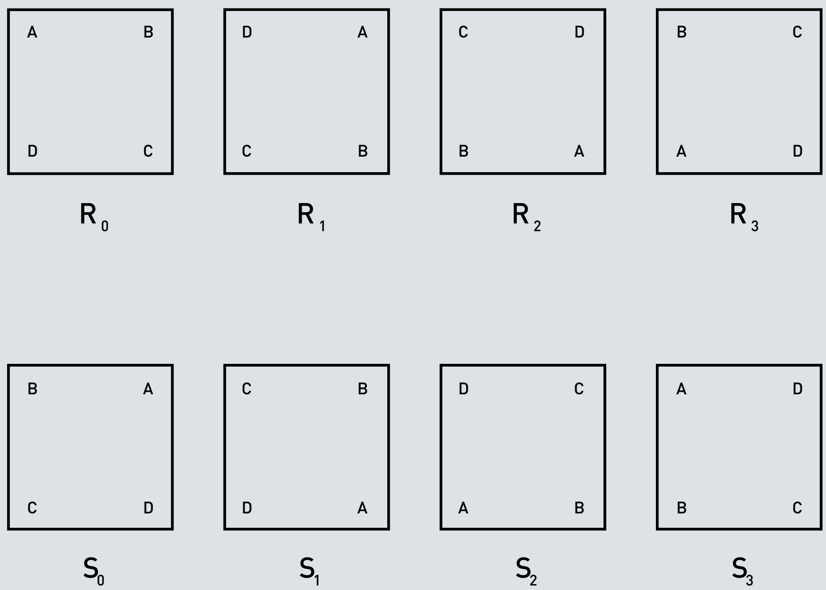
*	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_0								
R_1								
R_2								
R_3								
S_0								
S_1								
S_2								
S_3								

ACTIVITY 2

Worksheet 4 Square:

SYMMETRY TABLE

Answers:



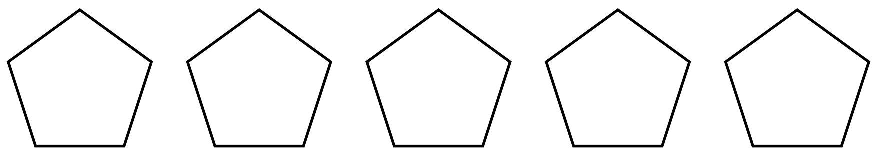
CAYLEY TABLE FOR EQUILATERAL TRIANGLE

*	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_0	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_1	R_1	R_2	R_3	R_0	S_1	S_2	S_3	S_0
R_2	R_2	R_3	R_0	R_1	S_2	S_3	S_0	S_1
R_3	R_3	R_0	R_1	R_2	S_3	S_0	S_1	S_2
S_0	S_0	S_3	S_2	S_1	R_0	R_3	R_2	R_1
S_1	S_1	S_0	S_3	S_2	R_1	R_0	R_3	R_2
S_2	S_2	S_1	S_0	S_3	R_2	R_1	R_0	R_3
S_3	S_3	S_2	S_1	S_0	R_3	R_2	R_1	R_0

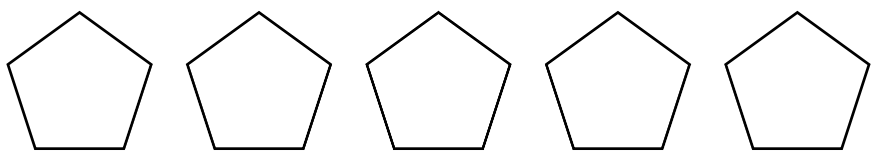
ACTIVITY 2

Worksheet 5 Regular Pentagon:

SYMMETRY TABLE



R_0 R_1 R_2 R_3 R_4



S_0 S_1 S_2 S_3 S_4

CAYLEY TABLE FOR REGULAR TRIANGLE

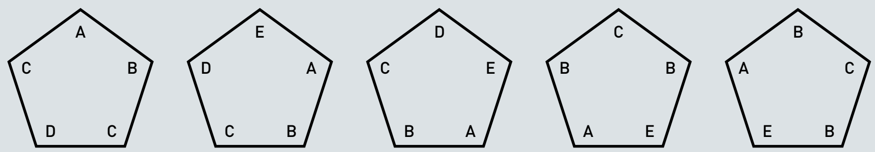
	R_0	R_1	R_2	R_3	R_4	S_0	S_1	S_2	S_3	S_4
R_0										
R_1										
R_2										
R_3										
R_4										
S_0										
S_1										
S_2										
S_3										
S_4										

ACTIVITY 2

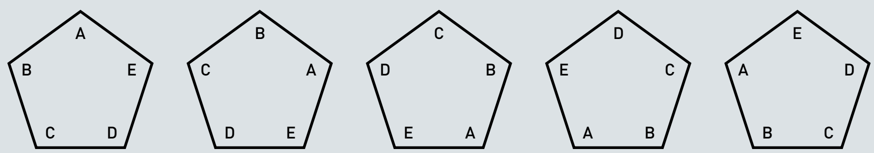
SYMMETRY TABLE

Worksheet 5 Regular Pentagon:

Answers:



R_0 R_1 R_2 R_3 R_4



S_0 S_1 S_2 S_3 S_4

CAYLEY TABLE FOR REGULAR TRIANGLE

	R_0	R_1	R_2	R_3	R_4	S_0	S_1	S_2	S_3	S_4
R_0	R_0	R_1	R_2	R_2	R_4	S_0	S_1	S_2	S_3	S_4
R_1	R_1	R_2	R_3	R_3	R_0	S_1	S_2	S_3	S_4	S_0
R_2	R_2	R_3	R_4	R_0	R_1	S_2	S_3	S_4	S_0	S_1
R_3	R_3	R_4	R_0	R_1	R_2	S_3	S_4	S_0	S_1	S_2
R_4	R_4	R_0	R_1	R_2	R_3	S_4	S_0	S_1	S_2	S_3
S_0	S_0	S_4	S_3	S_2	S_1	R_0	R_4	R_3	R_2	R_1
S_1	S_1	S_0	S_4	S_3	S_2	R_1	R_0	R_4	R_3	R_2
S_2	S_2	S_1	S_0	S_4	S_3	R_2	R_1	R_0	R_4	R_3
S_3	S_3	S_2	S_1	S_0	S_4	R_3	R_2	R_1	R_0	R_4
S_4	S_4	S_3	S_2	S_1	S_0	R_4	R_3	R_2	R_1	R_0

ACTIVITY 2

When you have completed the three tables, try to establish a set of rules that would allow you to construct a table for an *n*-gon:

1. For each table, what is $R_1 * R_2$?
Answer: Triangle: R_0. Square: R_3. Pentagon: R_3.

2. For the square and pentagon, what is $R_1 * R_3$?
Answer: Square: R_0. Pentagon: R_4.

3. For the pentagon, what is $R_1 * R_4$?
Answer: Pentagon: R_0.

4. What is the relationship between subscripts for combinations of rotations?

Hint: Think modular arithmetic.
Answer: The subscript of the resultant rotation is equal to the sum of the subscripts of the starting rotations, mod n. $R_i * R_j = R_{i+j} \text{ mod } n$

Facilitator's note: allow 5 – 10 minutes for the next question.

5. Find similar relationships for any reflection followed by any other reflection, reflections followed by rotations, and rotations followed by reflections.
(Again, think modular arithmetic.)

Encourage participants to look at specific examples of each case and make generalizations about how the subscripts combine. Remind them to think about arithmetic modulo n.

Answer: The rules for combining the rotations Rn and reflections Sn of an n-gon are as follows:

$R_i * R_j = R_{i+j \text{ mod } n}$ (already found in the previous question)
$R_i * S_j = S_{i+j \text{ mod } n}$
$S_i * R_j = S_{i-j \text{ mod } n}$
$S_i * S_j = R_{i-j \text{ mod } n}$

Reminder the participants that if i-j mod n is negative, it might be helpful to think of it in a mod cyclical form.

ACTIVITY 3

(50 minutes)

Facilitator's note: This activity is designed to follow Activity 2.

MATERIALS

- A couple of old pairs of pants
- A few shoeboxes (these are not essential, but they may prove helpful for people who have trouble visualizing the manipulation of 3-D objects in their heads)

The mathematical study of symmetry leads naturally to the identification and analysis of groups, which are sets of objects that obey the four specific rules outlined in the text. Group theory provides a way to find abstract general structure in a set of objects, whether they be motions that leave a geometric object invariant or the rules that dictate inter-clan marriage arrangements. In the following activity, you will get a sense of the range of situations that can be analyzed using group theory.

A (10 minutes)

The first group that you will explore is based on what can be done with a pair of pants (while you're not wearing them, of course). The elements of this group will be certain manipulations of the pants, as follows:

Motion X = Turn the pants around from back to front.
Motion Y = Turn the pants inside out.
Motion Z = Turn the pants inside out and back to front.
Motion I = Identity—leave the pants alone.

Since every group consists of both a set of elements and an operation, we need to define an operation for this group. The one we used in the previous activity "followed by" will work just fine. So "X followed by Y" means "turn the pants around from back to front and then turn them inside out." Notice that this leaves the pants as if you had only done Z to them. Therefore, we can say that X followed by Y equals Z.

ACTIVITY 3

1. Construct a Cayley table for the defined motions and operations.

Answer:

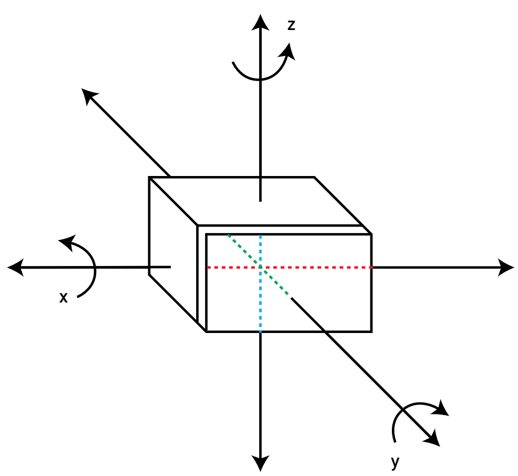
Followed by	X	Y	Z	I
X	I	Z	Y	X
Y	Z	I	X	Y
Z	Y	X	I	Z
I	X	Y	Z	I

2. Is this a group? Justify your answer.

Answer: There are four tests for a group: 1) closure; 2) existence of an inverse for each element; 3) existence of an identity element; and 4) associativity. Looking at the table, you can see that the set is closed under the operation, each element serves as its own inverse, there is an identity element, and associativity can be assumed or demonstrated with a couple of examples.

Convene the group to discuss the table and whether or not this counts as a group. (5 minutes)

B (20 minutes)



The second group to look at involves a certain set of motions with a shoebox.

ACTIVITY 3

The motions to look at are:

L = a 180-degree rotation about the x-axis
M = a 180-degree rotation about the y-axis
N = a 180-degree rotation about the z-axis
I = the identity—no rotations

Explore the behavior of these elements under the operation *, denoting "followed by." For example, L * M leaves the box in the same position it would have been in after the motion N. L * M = N.

1. Complete the operation table for this set and comment on whether or not it is a group under the operation "followed by."

Answer:

Followed by	L	M	N	I
L	I	N	M	L
M	N	I	L	M
N	M	L	I	N
I	L	M	N	I

This is a group;, the commentary should be similar to what was given in the previous problem.

C (20 Minutes)

Finally, let's look at group structure in the field of anthropology. The Kariera are a tribe of Australian Aboriginal people who have a complex set of rules that determine who can marry whom. The tribe consists of four clans: Banaka, Karimera, Burung, and Palyeri. The marriage rules are as follows:

A Banaka can only marry a Burung.
A Karimera can only marry a Palyeri.

ACTIVITY 3

For simplicity's sake, let's represent each clan with a letter:

A = Banaka
B = Karimera
C = Burung
D = Palyeri

1. Use the marriage rules to create a "family correspondence." Imagine an ordered set of four people, one from each clan, {A,B,C,D}; what would the set that comprises their spouses, in order, be?
Answer: {C,D,A,B}

2. Let f be the transformation of {A,B,C,D} to {C,D,A,B}. This means that if f acts on a set of four members, it replaces A with C, B with D, C with A, and D with B. What happens if you apply f to {A,B,C,D} twice in a row?
Answer: {A,B,C,D} → {C,D,A,B} → {A,B,C,D}; this action returns the set to its original order!

3. Now consider the Kariera rules for children.

The children of a male Banaka and a female Burung are Palyeri.
The children of a male Burung and a female Banaka are Karimera.
The children of a male Karimera and a female Palyeri are Burung.
The children of a male Palyeri and a female Karimera are Banaka.

Note that the clan of the child is determined by either the clan of the father or the clan of the mother, but a child is never of the same clan as the father or mother. Imagine an ordered set of the clans of the mothers, {A,B,C,D}. What will the ordered subset that corresponds to the clans of the children of the mothers look like?
Answer: {B,A,D,C}

4. Let m be the transformation that, given the subset of mothers' clans, {A,B,C,D}, outputs the subset of children's clans you found in the previous question. Describe what m does to {A,B,C,D}.
Answer: The transformation m turns A into B, B into A, C into D, and D into C.

ACTIVITY 3

5. What happens if you apply m to {A,B,C,D} twice in a row?
Answer: You get {A,B,C,D} again.

6. Let p be the transformation that, given the set of fathers' clans, {A,B,C,D}, outputs the set of the clans of their children. Describe what p does to {A,B,C,D} and give the resulting set of children's clans.
Answer: The transformation p turns A into D, B into C, C into B, and D into A. The result is {D,C,B,A}.

7. Start with {A,B,C,D}. Apply f;; then apply m to the result. What is the final result?
Answer: After f: {C,D,A,B}. After m: {D,C,B,A}.

8. Could the result of the sequence of two transformations that you did in the previous question be obtained using only one transformation? Which one?
Answer: Transformation p takes {A,B,C,D} and gives {D,C,B,A}.

9. Let * denote the operation "followed by," and let "I" be the identity transformation. Complete the following table:

Followed by	I	f	p	m
I				
f		I		p
p				
m				I

Answer:

Followed by	I	f	p	m
I	I	f	p	m
f	f	I	m	p
p	p	m	I	f
m	m	p	f	I

10. Is the set {I,f,m,p} a group under *? Explain.
Answer: Yes-the explanation is the same as that for the other tables in this exercise.

ACTIVITY 3

11. Look at the tables you have obtained for all three groups. If you let X = L = f, Y = M = p, and Z = N = m, what do you notice? What is going on here?
Answer: All three groups have the same operation table. They have the same group structure. Note that all the groups have the property that each element serves as its own inverse and there are four elements. This is the abstract structure that underlies all three groups.

Facilitator's note: You can connect this set of activities with the quote from the video, to the effect of "… one who thinks algebra and geometry are different is wrong…." After responding to question 10, participants might get this idea to some extent, and it could be this great moment that connects everything—the past, the art, and the math!

Convene the large group and discuss results. (5 minutes)

QUESTION FOR DISCUSSION:

Why do you think tables such as these are tools in what is called "abstract algebra?" What is the connection to what we normally call "algebra?"

ACTIVITY 4

(30 minutes)

Facilitator's note: This activity can stand alone from Activities 2 and 3.

In the previous activity, you looked at transformations that take an ordered set as input and output a permutation of that set. In the text, we saw how symmetry groups correspond to permutation groups, the symmetries of an equilateral triangle matching up with the permutations of a set of three objects, for example. This final activity deals with permutations without necessarily addressing an underlying group structure. It is actually more of an exercise in probability, but it is hopefully a fun one and one that may be of interest to students.

"How many people should a person date before deciding on the one?"

The problem of finding the right spouse or life partner is one that most people at least consider at some point in their lives. With the exceptions of arranged marriages and "Love at First Sight" scenarios, people tend to meet more than one person who is a suitable spouse or life partner, some more suitable than others. The question is: after meeting how many possible partners should you choose somebody?

First, a few simplifying assumptions:

- Assume that you will meet four potential partners in your life.
- Assume that you could rank these potential partners 1 through 4, with 4 being the most suitable and 1 being the least suitable.
- Assume that you can see only one potential partner at a time, and if you reject a person, that person can never be your partner.

1. How many possible "partner-meeting" orders are there? List them.

Answer: There are $4! = 4 \times 3 \times 2 \times 1 = 24$ possible orderings. They are:

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

ACTIVITY 4

2. What is the probability of meeting the most suitable partner on the first try in this scenario? Explain.

Answer: The most suitable partner is partner 4. There are six orderings that start with 4: 4321, 4312, 4231, 4213, 4132, and 4123. The probability is then $\frac{6}{24}$ or 0.25 that one will meet the optimal person on the first try.

3. What is the probability of meeting the best partner on the last try?

Answer: 0.25 again. The six permutations ending in 4 are: 3214, 3124, 2314, 2134, 1324, and 1234.

Facilitator's note: A clear understanding of the answers to questions 2 and 3 is essential to moving on. The step of counting up permutations that give the desired result and dividing by all possible permutations is sufficiently multi-tiered that it will be confusing to many without real clarity on 2 and 3.

Let's say that, as a general strategy, you meet and reject a defined number of people and then take the first person after that who is better than the ones previously rejected. For example, suppose that you decide to reject the first two possible partners and choose the first one after that who is better than either of those first two. The ordering 3214 then would work out this way: you reject persons 3 and 2 automatically and then meet and reject person 1 because this person is not better than 3 or 2. When you meet person 4, you stop, because 4 is better than everyone you have met so far (not to mention the fact that 4 is the last of the possibilities).

4. What are the chances that this strategy (automatically rejecting the first two) will result in you ending up with person 4?

Hint 1: Start by looking at all the orderings in which person 4 is not one of the first two people.

Answer: 1234, 1243, 1324, 1342, 2134, 2143, 2314, 2341, 3124, 3142, 3214, and 3241

ACTIVITY 4

Hint 2: How many of these orderings will result in picking person 4 according to the rule "reject the first two and pick the first one after that who is better than either of those first two?"

Answer: The orderings 1234 and 2134 will result in your picking person 3; the rest will result in your picking person 4. This leaves 10 possible orderings in which person 4 will be picked, so the probability of success with this strategy is $\frac{10}{24}$ or ~ 0.42.

5. Use a similar process to determine the probability of ending up with person 4 if you follow the rule: "reject the first person and then choose the next person after that who is better than the first."

Answer: The possible orderings in which the first person is not 4 are:

1234	2134	3124
1243	2143	3142
1324	2314	3214
1342	2341	3241
1423	2413	3412
1432	2431	3421

Of these, the following orderings correspond to choosing person 4:
3421, 3412, 3241, 3214, 3142, 3124, 2431, 2413, 2143, 1432, and 1423

So, $\frac{11}{24}$ or ~ 0.46 is the probability of ending up with person 4 using this method.

6. Of the strategies discussed, which has the highest chance of success (defined as ending up with person 4)?

Answer: The best strategy is to reject the first person and then choose the next person after that who is better than the first. It has a 46% chance of success, which is better than the 42% or 25% chances associated with the other strategies discussed.

7. What are the limitations of this model?

Answer: Answers will vary.

CONCLUSION

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS.

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 6, The Beauty of Symmetry could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: Making Sense of Randomness. Workshop participants are expected to read the accompanying text for Unit 7, Making Sense of Randomness before the next session.

NOTES