Session 9

Solids

Key Terms for This Session

Previously Introduced

- congruent
- regular polygon
- vertex

New in This Session

- cross section
- edge
- face
- net
- Platonic solid
- polyhedron

Introduction

In this session, you will build solids, including Platonic solids, in order to explore some of their properties. By creating and manipulating these solids, you will develop a deeper sense of some of the geometric relationships between them.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will do the following:

- Learn about various aspects of solid geometry
- Explore Platonic solids and why there is a finite number of them
- Examine two-dimensional properties of three-dimensional figures such as nets and cross sections

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Note 1.

Materials Needed:

- Polydrons or other snap-together regular polygons
- dental floss or piano wire
- a party hat
- clay
- molds for a cube, tetrahedron, cylinder, cone, and sphere (optional)
- scissors

Polydrons

You can purchase Polydrons from the following source:

Polydron International Limited
Tel: 0044 (0)1285 770055 • Fax: 0044 (0)1285 770171
www.polydron.com

An alternative to purchasing Polydrons is to make cutouts of regular polygons from stiff paper or poster board and use tape to attach them. Each individual working alone or pair of participants will need at least 32 triangles, 12 pentagons, six squares, and three hexagons, but it's helpful if each pair has extra sets of each kind of polygon.

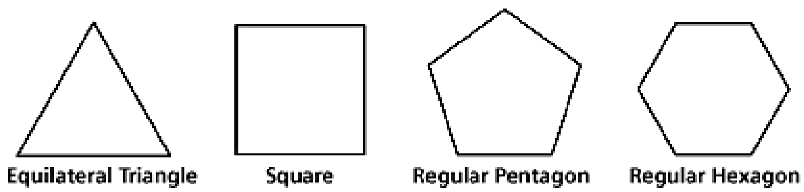
Part A: Platonic Solids (45 minutes)

Building the Solids

Platonic solids have the following characteristics:

- All of the faces are congruent regular polygons.
- At each vertex, the same number of regular polygons meet.

In order to do the following problems, you will need Polydrons or other snap-together regular polygons. If you don't have access to them, take out page 206 as a template for cutting shapes out of stiff paper or poster board.



Problem A1.

a. Connect three triangles together around a vertex. Complete the solid so that each vertex is the same. What do you notice? Were you able to build a solid?

b. Repeat the process with four triangles around a vertex, then five, then six, and so on. What do you notice?

Problem A2.

a. Connect three squares together around a vertex. Complete the solid so that each vertex is the same. What do you notice? Were you able to build a solid?

b. Repeat the process with four squares around a vertex, then five, and so on. What do you notice?

Problem A3.

a. Connect three pentagons together around a vertex. Complete the solid so that each vertex is the same. What do you notice? Were you able to build a solid?

b. Repeat the process with four pentagons, then five, and so on. What do you notice?

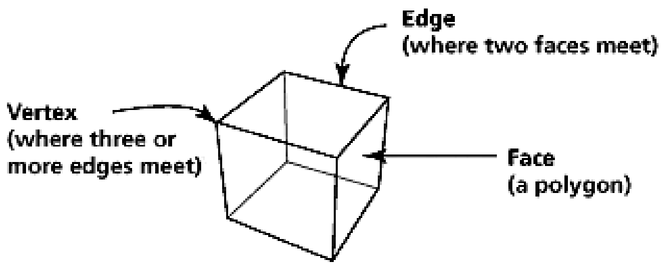
Problem A4. Connect three hexagons together around a vertex. Complete the solid so that each vertex is the same. What do you notice? Were you able to build a solid?

Problem A5. How many Platonic solids are there? Explain why that's the case.

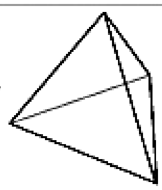
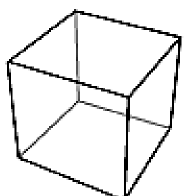
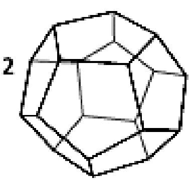
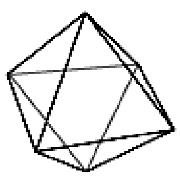
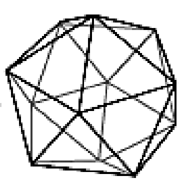
Part A, cont'd.

Naming the Solids

When working with three-dimensional figures, the terminology can get confusing. (What would you call a side?) It helps if everyone uses the standard names:



The Platonic solids are named for the number of faces they have.

Triangular Faces	Square Faces	Pentagonal Faces
4  Tetrahedron	6  Cube (Hexahedron)	12  Dodecahedron
8  Octahedron		
20  Icosahedron		

Part A, cont'd.

Take It Further

Problem A6. For each of the Platonic solids, count the number of vertices, faces, and edges. This is harder than it sounds! Think about how to "count them without counting them."

Solid	Vertices	Faces	Edges
Tetrahedron			
Octahedron			
Icosahedron			
Cube			
Dodecahedron			

[See Tip A6, page 207]

Problem A7. Find a pattern in your table. Express it as a formula relating vertices (*v*), faces (*f*), and edges (*e*).

Problem A8. Does your pattern hold for solids other than the Platonic solids? Build several other solids. Count the vertices, faces, and edges, and find out!

Video Segment (approximate time: 5:40-7:43): You can find this segment on the session video approximately 5 minutes and 40 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

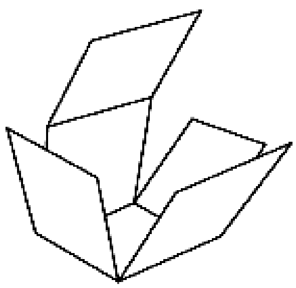
In this video segment, the participants build solids and start to make conjectures about the relationship between angles, edges, and faces needed to build them. Watch this segment after you've completed Problems A6-A8.

How many solids were you able to build? Were you convinced that there are only five Platonic solids?

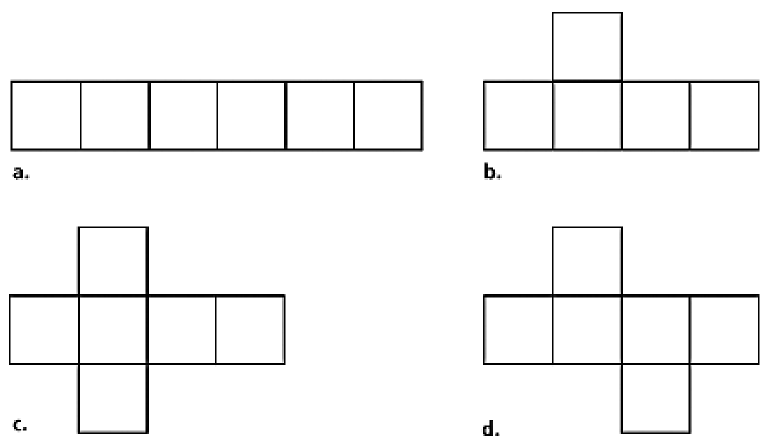
Part B: Nets (30 minutes)

Nets With Regular Polygons

Imagine unfolding a cube so that all of its faces are laid out as a set of squares attached at their edges. The resulting diagram is called a net for a cube. There are many different nets for a cube, depending on how you unfold it. **[See Note 2]**



Problem B1. Which of the following are nets for a cube? Explain how you decided. Try to imagine folding each one, or cut them out to explore.



Problem B2. How many different nets for a cube can you make? How did you think about the problem, what method did you use to generate different nets, and how did you check whether or not a new one really was different?

Video Segments (approximate times: 16:52-18:06 and 18:29-19:55): You can find the first part of this segment on the session video approximately 16 minutes and 52 seconds after the Annenberg/CPB logo. The second part begins 18 minutes and 29 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

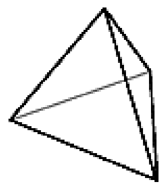
Modeling objects in three dimensions as well as finding their two-dimensional representation is an important part of geometric thinking. Additionally, finding all possible nets of an object can be a challenging task in terms of ability to organize and record your testing methods. After you've completed Problems B1 and B2, watch this sequence to further explore different reasoning participants used to find all the nets for a cube.

What method did you come up with to organize and record different nets?

Note 2. If you still have a cube built from polydrons, you can unfold it, but leave all the squares connected to each other. If you are working in a group, compare how everyone unfolded their own cubes to see if others unfolded them the same way.

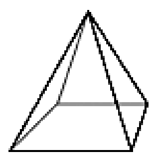
Part B, cont'd.

Problem B3. Find all the possible nets for a regular tetrahedron.



Take It Further

Problem B4. A square pyramid has a square base and triangular faces that meet at a top vertex. Find all the possible nets for a square pyramid.



Nets With Circles

Problem B5. What would a net for a cylinder look like? Sketch one, and describe how the various edges are related.

Problem B6. A cone has a circular base and a point at the vertex. If you cut open a cone and unrolled it, what do you think you would have as the net? Predict what the net for a cone will look like, and then draw a picture of your prediction.

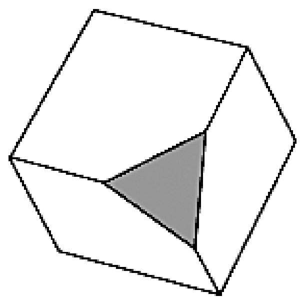
Problem B7. Take a party hat or some other cone-shaped object (a "right" cone where the vertex is directly above the center of the circle). Make a single slice up the side of the cone, and unroll it. What shape do you have? Explain why that is the correct net for a cone.

Problem B8. If you took a larger sector of the same circle, how would that change the cone? What if you took a smaller sector?

Problems B1-B4 are adapted from *Connected Geometry,* developed by Educational Development Center, Inc. www.glencoe.com/sec/math

Part C: Cross Sections (45 minutes)

A cross section is the face you get when you make one slice through an object. Below is a sample slice through a cube, showing one of the cross sections you can get.



The polygon formed by the slice is the cross section. The cross section cannot contain any piece of the original face; it all comes from "inside" the solid. In this picture, only the gray piece is a cross section.

To work on Problems C1 to C4, you may want to use clay solids and dental floss to derive your answers. Alternatively, you may want to use colored water in plastic solids. **[See Note 3]**

> **Try It Online!** **www.learner.org**
>
> Problems C1 and C2 can also be explored as an Interactive Activity. Go to the Geometry Web site at www.learner.org/learningmath and find Session 9, Part C.

Problem C1. Try to make the following cross sections by slicing a cube:

a. a square

b. an equilateral triangle

c. a rectangle that is not a square

d. a triangle that is not equilateral

e. a pentagon

f. a hexagon

g. an octagon

h. a parallelogram that is not a rectangle

i. a circle

Record which of the shapes you were able to create and how you did it. The Interactive Activity provides you with one way to make each of the shapes that you can, in fact, make as a cross section. **[See Tip C1, page 207]**

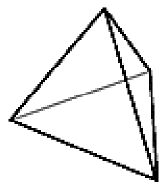
Note 3. To use clay solids and dental floss to derive your answers, mold a cube from the clay. Then use the dental floss or piano wire to make a straight slice through the cube, creating a cross section. When making a cut, you may end up with "bent" cuts, but what you want to try to make is a planar slice. For example, think about slicing straight through a loaf of bread.

Alternatively, you may want to use colored water in plastic solids. Fill the plastic solids part way, and then tilt them at different angles to see the faces of cross sections created by an imaginary slice along the surface of the water.

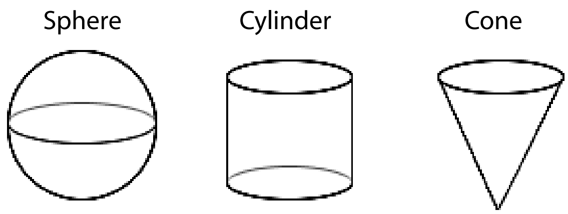
Part C, cont'd.

Problem C2. A couple of the shapes on the list in Problem C1 are impossible to make by slicing a cube. Explain what makes them impossible.

Problem C3. Find a way to slice a tetrahedron to make a square cross section. How did you do it?



Problem C4. What cross sections can you get from each of these figures?

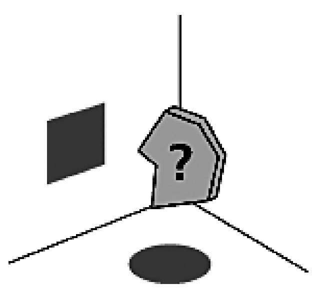


Polygon Shadows

You've looked at some of the "two-dimensional" properties of three-dimensional figures: their surfaces (nets) and their cross sections. Solid figures also cast two-dimensional shadows. Do you think you can recreate a solid by knowing what shadows it casts?

Take It Further

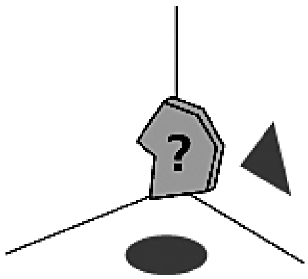
Problem C5. A solid object casts a circular shadow on the floor. When it is lit from the front, it casts a square shadow on the back wall. Try to build the object out of clay. Can you name it?



Part C, cont'd.

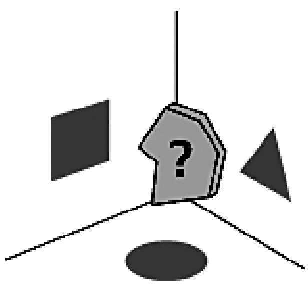
Take It Further

Problem C6. A solid object casts a circular shadow on the floor. When it is lit from the left, it casts a triangular shadow on the back wall. Try to build the object out of clay. Can you name it?



Take It Further

Problem C7. Suppose the object casts a circular shadow on the floor, a square shadow when lit from the front, and a triangular shadow when lit from the left. Try to build the object out of clay. Can you name it? **[See Note 4]**



Homework

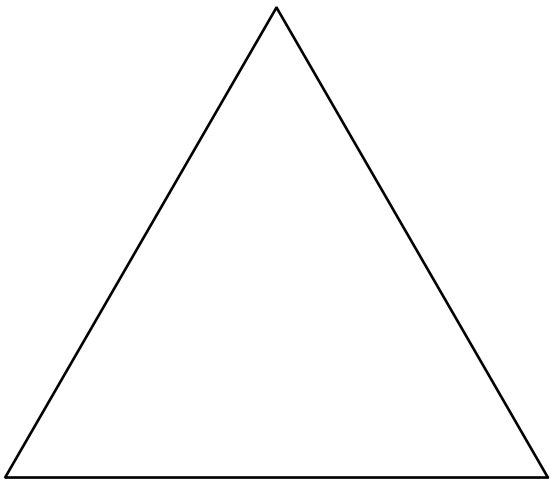
Suggested Reading

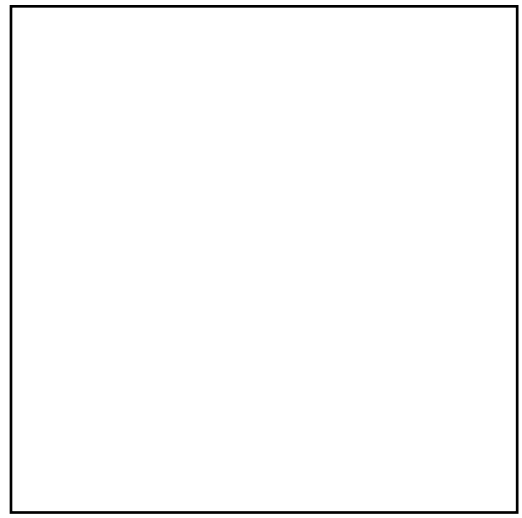
This reading is available as a downloadable PDF file on the *Geometry* Web site. Go to www.learner.org/learningmath.

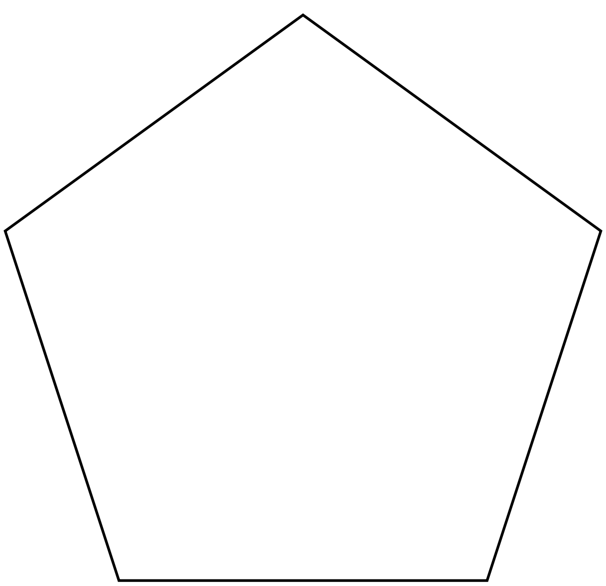
Van de Walle, John A. (2001). Geometric Thinking and Geometric Concepts. In *Elementary and Middle School Mathematics: Teaching Developmentally, 4th ed.* (pp. 342-349). Boston: Allyn & Bacon.

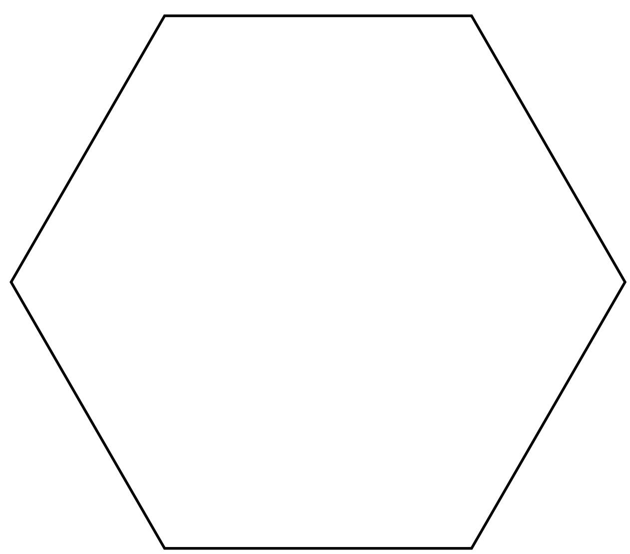
Note 4. You may want to further explore Problems C5-C7 on your own using a flashlight and objects described in solutions to these problems.

Polygons









Tips

Part A: Platonic Solids

Tip A6. Counting vertices and edges can be tricky. Think about how to "count without counting." How many faces are there on the polyhedron? How many vertices are there on each face? How many faces meet at a vertex on the polyhedron? You can put all of this information together to "count" the number of vertices.

Part C: Cross Sections

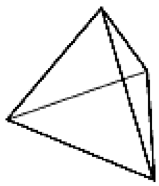
Tip C1. How may faces does a cube have? Each side of your cross section comes from cutting through a face of your cube.

Solutions

Part A: Platonic Solids

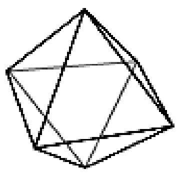
Problem A1.

a. Three triangles around a vertex: You will get a figure with four triangular faces. It is called a tetrahedron.



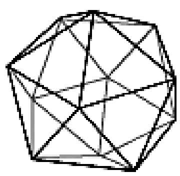
tetrahedron

b. Four triangles around a vertex: You will get a figure with eight triangular faces. It is called an octahedron.



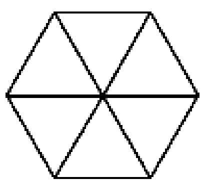
octahedron

Five triangles around a vertex: You will get a figure with 20 triangular faces. It is called an icosahedron.



icosahedron

Six triangles around a vertex: It lies flat, and you can't make a solid.



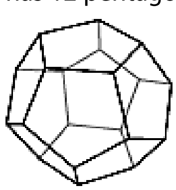
Problem A2.

a. Follow the instructions. The construction works, and you can construct a cube. A cube uses a total of six squares in the construction.

b. It is impossible to make a solid if we try to connect four or more squares at a common vertex.

Problem A3.

a. Follow the instructions. The construction works, and you can construct a dodecahedron. A dodecahedron has 12 pentagon faces.



b. It is impossible to make a solid if we try to connect four or more regular pentagons at a common vertex.

Solutions, cont'd.

Problem A4. It is impossible to create a solid by connecting regular hexagons at a common vertex. If you have ever seen a soccer ball, you may have noticed that it is made up of hexagons and pentagons, but not of hexagons alone!

Problem A5. There are only five Platonic solids. The reason is that the sum of the interior angles of the regular polygons meeting at a vertex must not equal or surpass 360°. Otherwise, the figure will lie flat or even fold in on itself.

We need at least three faces to meet at a vertex, but after hexagons, the regular polygons all have interior angles that are more than 120°. So if three of them met at a vertex, there would be more than 360° there. We have already seen that this same angle restriction leads to just three possibilities for triangles (there must be at least three triangles but fewer than six), squares, and pentagons (in each case, there must be at least three but fewer than four).

This leaves a total of just five possible solids that fit the definition of "regular" or "Platonic" solid.

Problem A6.

Solid	Vertices	Faces	Edges
Tetrahedron	4	4	6
Octahedron	6	8	12
Icosahedron	12	20	30
Cube	8	6	12
Dodecahedron	20	12	30

Problem A7. The pattern emerging from each row of the table is that, if f is the number of faces, e is the number of edges, and v is the number of vertices, we have $v + f = e + 2$. (Note that there are several equivalent ways of writing this relationship!)

Problem A8. It turns out that $v + f = e + 2$ holds for all convex polyhedra. This is known as Euler's formula, named for one of the most famous mathematicians of all time; he created the formula and proved that it always held.

Part B: Nets

Problem B1.

a. This is not a net for a cube since it would not close.

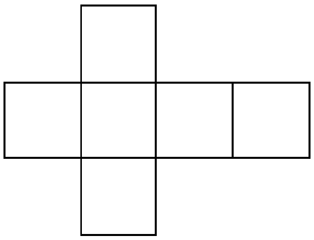
b. This is not a net for a cube since there are not enough faces.

c. Yes. Fold and check.

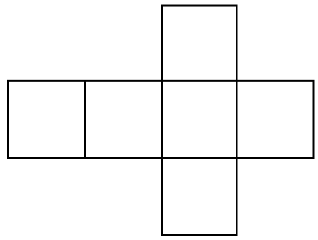
d. Yes. Fold and check.

Solutions, cont'd.

Problem B2. First we declare two nets identical if they are congruent; that is, they are the same if you can rotate or flip one of them and it looks just like the other. For example:



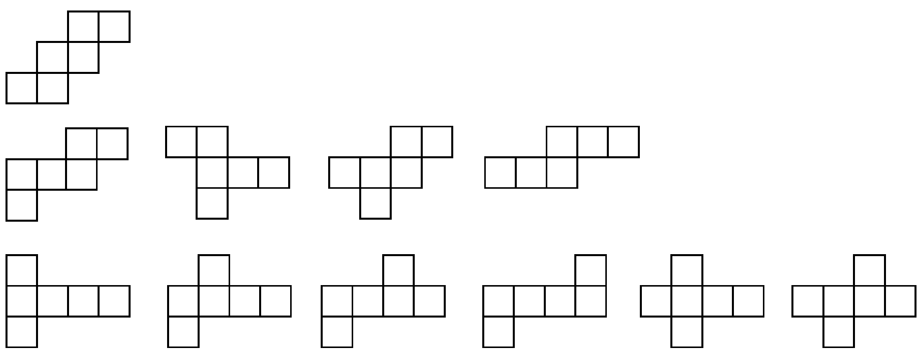
and



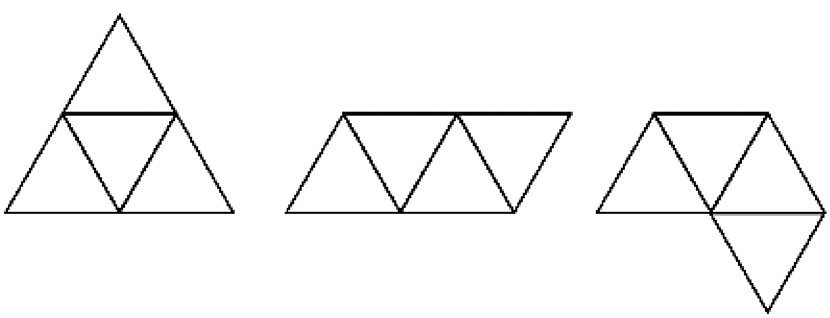
These can be considered identical since a 180° rotation turns one onto the other.

One way of classifying the nets is according to the number of squares aligned (four in the example above).

If five or six squares are aligned, we cannot fold the net into a cube since at least two squares would overlap and the cube would not be closed. So valid nets are to be found among those nets that have four, three, or two squares aligned. Eliminating redundancies (i.e., taking just one net from each pair of equivalent nets), we can come up with the following 11 valid nets:

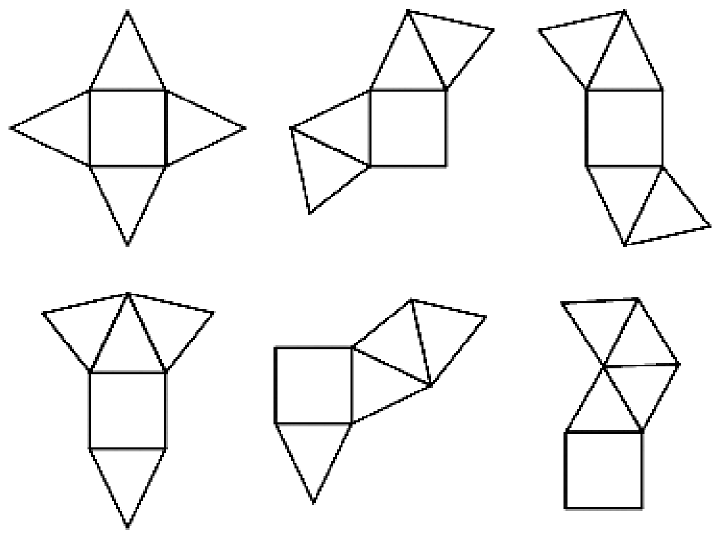


Problem B3.

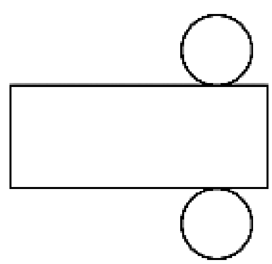


Solutions, cont'd.

Problem B4. If we assume that two nets are equivalent if one can be rotated or flipped to overlap with the other one, there are six possible nets for a square pyramid.



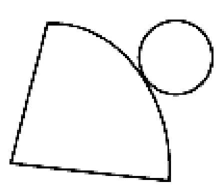
Problem B5. A net for a cylinder might look like this:



It will consist of a rectangle and two congruent circles. One pair of the rectangle's sides must have the same length as the circumference of the two circles. The two circles must attach to those two sides of the rectangle, though they need not be positioned exactly opposite each other as shown here.

You can test this by unfolding a layer from a roll of paper towels. Results differ from the rectangle most people will predict.

Problem B6. Predictions may vary. Many people are surprised to find that the net will look like a circle and a sector of a larger circle:

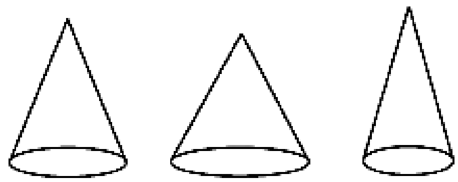


As with the cylinder, the circumference of the circle must equal the length of the arc of the given sector.

Problem B7. The shape we get is a sector of a circle. This is because every point on the bottom edge of the cone-shaped object or party hat is equidistant from the top point. This is also true of a sector of a circle because all the radii of the same circle are the same length.

Solutions, cont'd.

Problem B8. A larger sector would increase the area of the base and decrease the height of the cone, while a smaller sector would decrease the area of the base and increase the height. All the radii of the same circle are the same length.

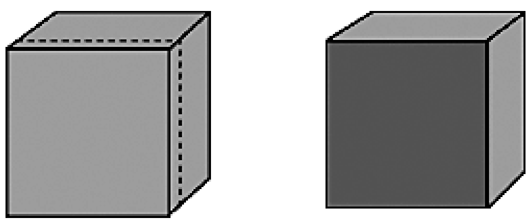


Note that you can test this out with the party hats: You can cut a piece off one of them from the center to its edge, then refold it and compare it to the original. You can also tape together two unfolded party hats of the same size and then refold to see what kind of cone (hat) you get.

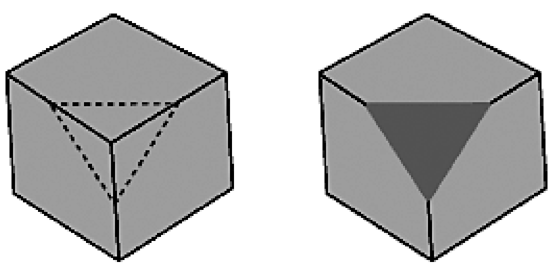
Part C: Cross Sections

Problem C1.

a. A square cross section can be created by slicing the cube by a plane parallel to one of its sides.



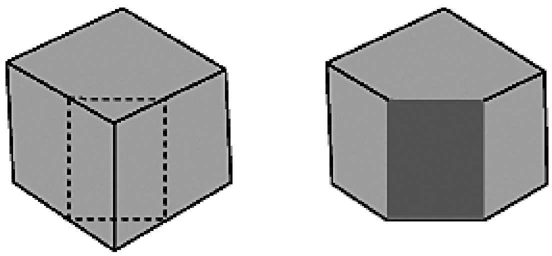
b. An equilateral triangle cross section can be obtained by cutting the cube by a plane defined by the mid-points of the three edges emanating from any one vertex.



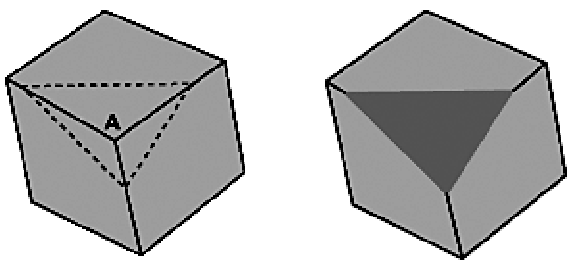
Solutions, cont'd.

Problem C1, cont'd.

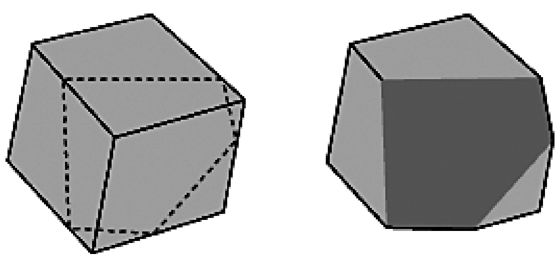
c. One way to obtain a rectangle that is not a square is by cutting the cube with a plane perpendicular to one of its faces (but not perpendicular to the edges of that face), and parallel to the four, in this case, vertical edges.



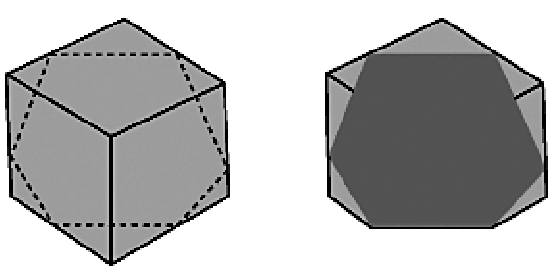
d. Pick a vertex, let's say A, and consider the three edges meeting at the vertex. Construct a plane that contains a point near a vertex (other than vertex A) on one of the three edges, a point in the middle of another one of the edges, and a third point that is neither in the middle nor coinciding with the first point. Slicing the cube with this plane creates a cross section that is a triangle, but not an equilateral triangle; it is a scalene triangle. Notice that if any two selected points are equidistant from the original vertex, the cross section would be an isosceles triangle.



e. To get a pentagon, slice with a plane going through five of the six faces of the cube.



f. To get a hexagon, slice with a plane going through all six faces of the cube.

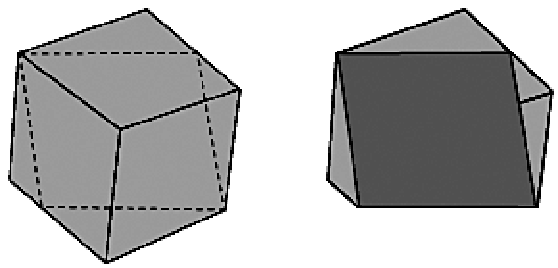


Solutions, cont'd.

Problem C1, cont'd.

g. It is not possible to create an octagonal cross section of a cube.

h. To create a non-rectangular parallelogram, slice with a plane from the top face to the bottom. The slice cannot be parallel to any side of the top face, and the slice must not be vertical. This allows the cut to form no 90˚ angles. One example is to cut through the top face at a corner and a midpoint of a non-adjacent side, and cut to a different corner and midpoint in the bottom face.

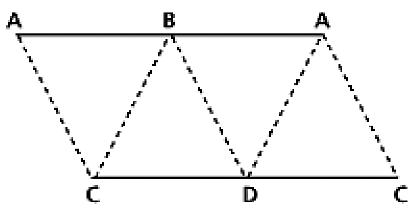


i. It is not possible to create a circular cross section of a cube.

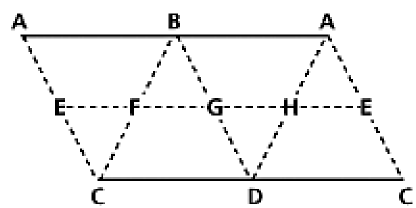
Problem C2. Whenever we cut the cube with a plane, each edge of the cross section corresponds to an intersection of one of the cube's faces with the plane. Since the cube has only six faces, it is impossible to cut it with one plane and create an octagonal cross section. Also, since the cube has no curved faces, a plane will not be able to intersect a cube and create a cross section with a curved segment in its perimeter.

Problem C3. One way to create a square cross section in a tetrahedron is to cut at the midpoints of four edges.

Alternatively, you can start with a net for the tetrahedron such as:



We then connect the midpoints of the sides with segments of equal length: EF, FG, GH, and HE.



When we fold the net into a tetrahedron, the points E, F, G, and H are on the same plane, and they define a square cross section when that plane cuts the tetrahedron.

Problem C4.

a. Any cross section of a sphere will be a circle.

b. Possible cross sections are circles (cut parallel to the base), rectangles, and ellipses.

c. Possible cross sections are circles (cut parallel to the circular base), ellipses (cut at an angle, not parallel to the circular base and not intersecting the base of the cone), parabolas (cut parallel to the edge of the cone, not intersecting the vertex but intersecting the base), and hyperbolas (cut perpendicular to the base, but not intersecting the vertex).

Problem C5. A right square cylinder, i.e., a cylinder whose height equals the diameter of its base.

Problem C6. A right circular cone.

Problem C7. It's a solid that looks like a "triangular" filter for a coffee maker, or the head (not handle) of a flathead screwdriver (pictured to the right).