Session 8

Similarity

Key Terms for This Session

Previously Introduced

- distance formula
- side-side-side (SSS) congruence
- triangle inequality

New in This Session

- angle-angle-angle (AAA) similarity
- cosine
- side-angle-side (SAS) similarity
- side-side-side (SSS) similarity
- similar
- sine
- tangent

Introduction

Similarity is one of the "big ideas" in geometry. Note that two things may be similar in colloquial English, but it is a much stronger statement to say that they are mathematically similar.

In this session, you will build on your intuitive notions of what makes a "good copy" to build a more formal definition of similarity. You will then look at applications of similar triangles, including triangle trigonometry.

For information on required and/or optional material for this session, **see Note 1.**

Learning Objectives

In this session you will learn to do the following:

- Explore geometric similarity as "reasoning about proportions"
- Study similar triangles
- Explore some basic ideas in trigonometry

Note 1.

Materials Needed:

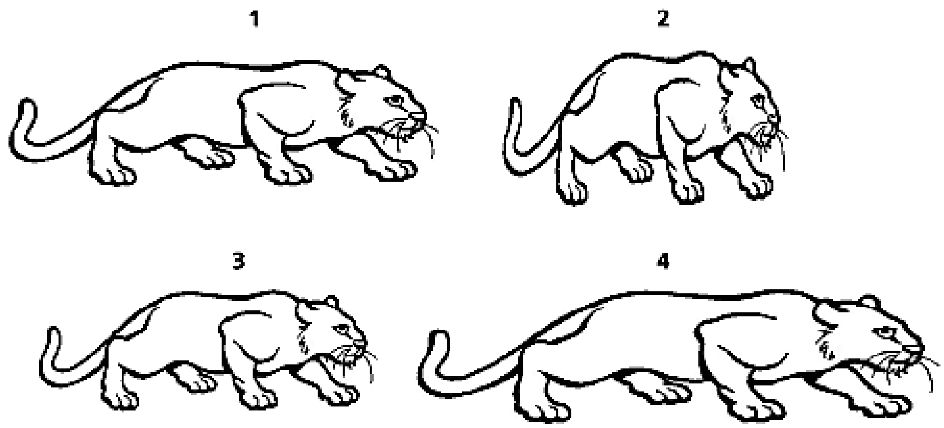
- graph paper
- blank white paper
- yardstick

Part A: Scale Drawings (50 minutes)

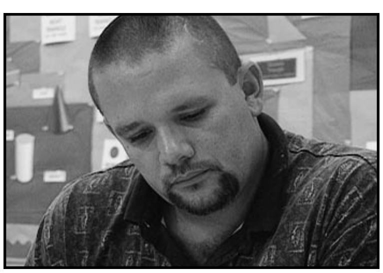
Finding a "Good Copy"

Here is a picture of a panther:

Problem A1. Which of these pictures is a good copy of the original picture? Explain how you made your decision.



[See Tip A1, page 191]



Video Segment (approximate time: 3:03-5:36): You can find this segment on the session video approximately 3 minutes and 3 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants explore different ideas that will help them decide which copy is the best copy of the original picture. Watch this segment after you've completed Problem A1.

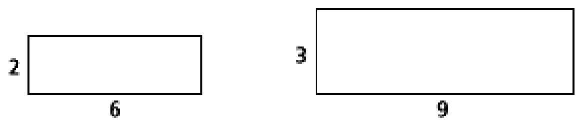
Compare your method with those the participants used in the video.

Part A, cont'd.

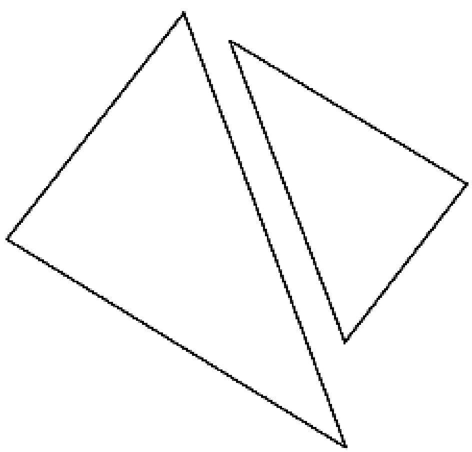
Mathematicians use the word "similar" to describe two figures that are the same shape but not necessarily the same size. If one is a scaled copy of the other, then two figures are similar.

Two polygons are similar if and only if corresponding angles have the same measure and corresponding sides are in proportion.

Problem A2. Are these two rectangles similar? How do you know?



Problem A3. Are these two triangles similar? How did you decide?

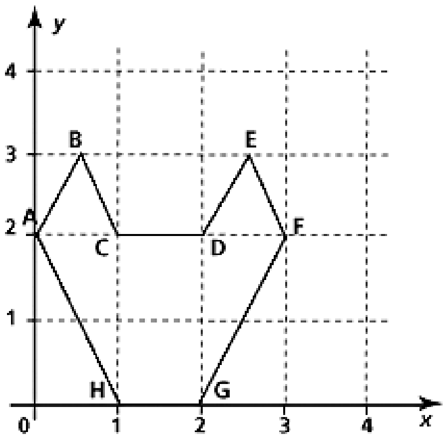


The Scale Drawings section is adapted from *Connected Geometry*, developed by Educational Development Center, Inc., © 2000 Glencoe/McGraw-Hill. Used with permission. www.glencoe.com/sec/math

Part A, cont'd.

Doubling the Coordinates

Here's a picture of a cat's head (it's really a polygon!) on a coordinate grid:



Problem A4.

a. Copy the picture of the cat's head onto a piece of graph paper, and find the coordinates of each point.

b. Multiply the coordinates of each point by 2, creating points A' through H'. Use the rule $(x,y) \mapsto (2x,2y)$.

c. Plot these new points on the same grid, and connect them in the same order.

d. Is your new figure similar to the original picture? How do you know?

Problem A5. Explain why, if you want to create a similar figure with sides twice as long as the original, you don't double all the angles as well.

The ratio between corresponding sides of the cat's head is called the ratio of proportionality. Since we made the sides twice as long, the ratio of proportionality is 2. **[See Note 2]**

Distances and Angles

Why does the coordinate trick work to create similar figures?

Rather than looking at the head of a cat, let's look at a triangle to explain why multiplying the coordinates by 2 creates a similar figure. We need to show two things:

1. The distance between any two points doubles.
2. The angles all stay the same.

Since we know that we can take any polygon and split it up into triangles, this will be enough to show that the scaling trick works for any polygon. If the triangles in the new polygon are all similar to the triangles in the original, then the two polygons themselves must be similar.

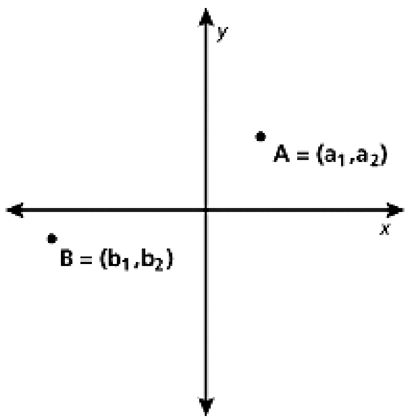
Note 2. In the cat's-head problem, you are doing a transformation that's called dilation. You've seen transformations before—such as translation, rotation, and reflection. These "rigid transformations" preserve congruence; when you apply one to a figure, it may move around, but it maintains its size and shape.

Dilation (scaling) can change not just the position of the figure, but also its size. There's another non-rigid transformation called a shear that can even change the overall shape of a figure—the final product, for example, may have different angle measurements than the original.

Problem A4 is adapted from *IMPACT Mathematics, Course 3*, developed by Educational Development Center, Inc., p. 329. © 2000 Glencoe/McGraw-Hill. Used with permission. www.glencoe.com/sec/math

Part A, cont'd.

Here are two arbitrary points, A = (a_1,a_2) and B = (b_1,b_2), on a coordinate grid:



Problem A6. What is the distance between the two points? (Look back to Session 6 if you don't remember how to find the distance. It might also help to try some specific examples.)

Problem A7. If you apply the rule (x,y) |→ (2x,2y) to both points to create A′ and B′, what are the new coordinates? What is the distance between A′ and B′? Are you sure that it's double the distance from A to B?

The reasoning that the angles don't change is a little trickier. Here's the idea: There is some triangle DEF, similar to triangle ABC, but with sides twice as long. (Imagine putting ABC in a copy machine and enlarging it by 100%.) The corresponding angles all have the same measures: m∠A = m∠D, m∠B = m∠E, and m∠C = m∠F.

Because triangles are rigid (SSS congruence), any triangles with the same sides as DEF will also have the same angles as DEF.

In particular, the triangle made by doubling the coordinates of A, B, and C has sides twice as long as triangle ABC. That is, the sides are the same as DEF. So the angles are also the same as DEF and as ABC.

That means the sides are in proportion to the sides of ABC (they're twice as long) and the angles have the same measures. So the two triangles are similar.

This argument is what we call general in principle. We explained why doubling the coordinates produces similar triangles. But the same argument would work if you multiplied the coordinates by any number.

Take It Further

Problem A8. How could you modify the definition of similar polygons to test whether figures like the two below are similar?

Part B: Similar Triangles (35 minutes)

The Mirror Trick

A mathematics teacher likes to astound her students with tricks that can be explained through mathematics. Before the class studies similarity, the teacher brings a mirror to class and performs this trick: **[See Note 3]**

The teacher puts the mirror on the floor facing up and asks a student to stand two feet from it. The teacher then positions herself so that she can just see the top of the student's head when she looks in the mirror. With a quick calculation, she reports the student's height. She's able to do the trick on every student in class.

Problem B1. Sketch the teacher, student, and mirror. Find a pair of similar triangles in your sketch, and explain how the teacher does her mirror trick. **[See Note 4]** **[See Tip B1, page 191]**

Video Segment (approximate time: 11:27-13:49): You can find this segment on the session video approximately 11 minutes and 27 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Similarity of triangles is frequently used in indirect measurement. In this segment, the participants watch a demonstration of the mirror trick and see how similarity of triangles can be used to estimate height. Watch this segment after you've completed Problem B1.

Think about why in this situation having the same angle measurement is enough to make two triangles similar.

Similarity Tests

Problem B2. For each pair of angles given below, sketch at least three different triangles that have these two angle measures. What do you notice? **[See Note 6]**

a. 90° and 60°

b. 45° and 45°

c. 120° and 30°

d. 80° and 40°

In Part A, we outlined an argument that if two triangles have all three pairs of sides in proportion, the triangles must be similar. This is the SSS similarity test.

Note 3. If you are working in a group, you may want to start by actually doing the "mirror trick" described here. When you calculate, remember that you should use a number closer to the height of your eyes rather than your actual height, but position yourself so that you just see the top of the other person's head. For Problem B1, you can ask others to figure out your height based on the setup used and the other person's (now known) height.

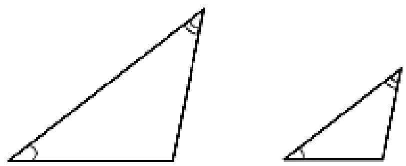
Note 4. If you are working in a group, compare the triangles you made with the triangles other people made and discuss what you notice.

Note 6. If you're working in a group, have each participant create just one triangle for each part, and then compare the results as a group.

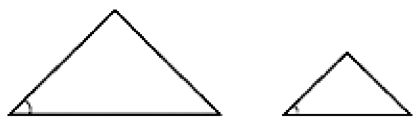
Part B, cont'd.

There are two other similarity tests:

- **AAA similarity:** If two triangles have corresponding angles that are congruent, then the triangles are similar. Because the sum of the angles in a triangle must be 180°, you really only need to know that two pairs of corresponding angles are congruent to know the triangles are similar.

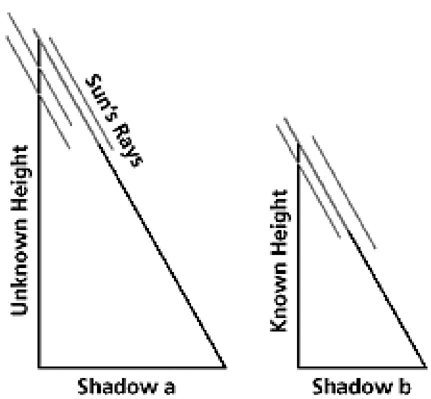


- **SAS similarity:** If two triangles have two pairs of sides that are proportional and the included angles are congruent, then the triangles are similar.



Measuring With Shadows

Using shadows is a quick way to estimate the heights of trees, flagpoles, buildings, and other tall objects. To begin, pick an object whose height may be impractical to measure, and then measure the length of the shadow your object casts. Also measure the shadow cast at the same time of day using a yardstick (or some other object of known height) standing straight up on the ground. Since you know the lengths of the two shadows and the length of the yardstick, you can use the fact that the sun's rays are approximately parallel to set up a proportion with similar triangles.



Because the sun's rays are parallel, the triangles are similar. Thus:

$$\frac{\text{Unknown Height}}{\text{Shadow } a} = \frac{\text{Known Height}}{\text{Shadow } b}$$

Part B, cont'd.

Problem B3. On a sunny day, Michelle and Nancy noticed that their shadows were different lengths. Nancy measured Michelle's shadow and found that it was 96 inches long. Michelle then measured Nancy's shadow and found that it was 102 inches long.

a. Who do you think is taller, Nancy or Michelle? Why?

b. If Michelle is 5 feet 4 inches tall, how tall is Nancy?

c. If Nancy is 5 feet 4 inches tall, how tall is Michelle?

Video Segment (approximate times: 21:10-21:44 and 22:23-25:13): You can find the first of these segments on the session video approximately 21 minutes and 10 seconds after the Annenberg/CPB logo. The second part begins approximately 22 minutes and 23 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In today's professional world, there are many practical applications that rely on triangle similarity. One such application is in medicine, where similar triangles are used to calculate the position of radiation treatment for cancer patients. In this segment, Sandra John-Baptiste and Jason Talkington work with dosimetrist Max Buscher to calculate the position of radiation beams so as to avoid their overlap on the spinal cord.

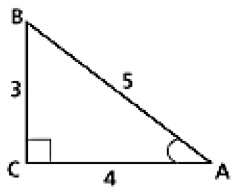
Can you think of any other applications of triangle similarity?

Part C: Trigonometry (35 minutes)

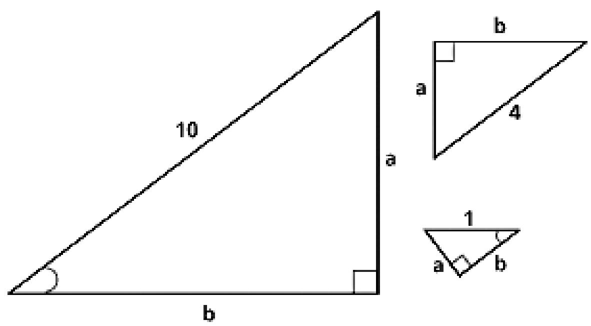
Right Triangle Ratios

The word "trigonometry" is enough to strike fear into the hearts of many high school students. But it simply means "triangle measuring." A trigon is another name for a triangle—think pentagon and hexagon, for example—and "-metry" means "measuring," just as it does in "geometry," or "earth measuring." Trigonometry is about measuring similar right triangles.

Problem C1. Here's a right triangle:



The triangles below are all similar to the original triangle above. Find the length *a* for each triangle. Explain how you did it. **[See Note 7]**



[See Tip C1, page 191]

Problem C2. For each triangle above, find the length *b*. Explain how you did it.

If you solved the problems above, you probably used the sine and cosine functions, even if you didn't know you were doing it. To find the missing value of *a,* you need to multiply the length of the hypotenuse by 3/5. To find the length *b,* you multiply the hypotenuse by 4/5.

Those ratios—(side *a*)/(hypotenuse) = 3/5 and (side *b*)/(hypotenuse) = 4/5—will hold for any right triangle that has another angle the same as A. All such right triangles will be similar to each other, so the ratios must be the same as in our original triangle.

Note 7. If you're working with a group, compare different methods other participants have come up with. Think about how these methods connect to yours.

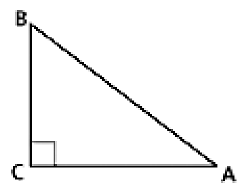
Part C, cont'd.

Trigonometric Functions

As you've seen, the trigonometric functions such as sine, cosine, and tangent are nothing more than ratios of particular sides in right-angle triangles. **[See Note 8]** These ratios depend only on the measure of an angle and have special names:

$$\frac{\text{Length of Opposite Side}}{\text{Length of Hypotenuse}} = \text{Sine of the Angle} \qquad \frac{\text{Length of Adjacent Side}}{\text{Length of Hypotenuse}} = \text{Cosine of the Angle}$$

$$\frac{\text{Length of Opposite Side}}{\text{Length of Adjacent Side}} = \text{Tangent of the Angle}$$

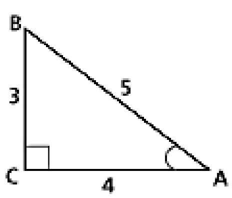


In the triangle above, the sine of angle A is defined as BC/AB. It is abbreviated sin A.

The cosine of angle A is defined as AC/AB. It is abbreviated cos A.

The tangent of angle A is defined as BC/AC. It is abbreviated as tan A. **[See Note 9]**

Problem C3. For the original triangle:



a. Find sin A and cos A.

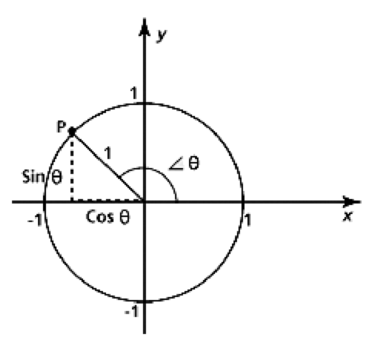
b. Find sin B and cos B.

Note 8. These ratios have been immensely useful in many fields, such as engineering, surveying, astronomy, and architecture. In addition to the three trigonometric functions mentioned in this session (sine, cosine, and tangent), there are three additional functions not covered in this course (secant, cosecant, and cotangent).

Note 9. In mathematics, if we have an idea that works in some particular cases, we often look for ways to extend that idea to a more general situation. For example, the definitions of the trigonometric functions on the right triangle are valid only when dealing with angles between 0º and 90º—no other angle can appear in a right triangle!

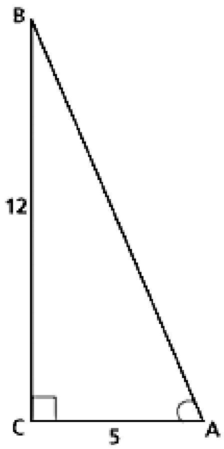
So, in order to make sense of and compute the values of trigonometric functions of any angle, we need to extend this definition. According to the extended definition, sin and cos of angle θ (theta) are defined to be *y* and *x* coordinates, respectively, of a point on the unit circle. (The unit circle is a circle centered at the origin of the coordinate system, with a radius equal to 1.)

Notice that as we move the point P along the circle to create angles between 0º and 360º, some coordinates will be positive and some will be negative.



Part C, cont'd.

Problem C4. For the following triangle:



a. Find sin A and cos A.

b. Find sin B and cos B.

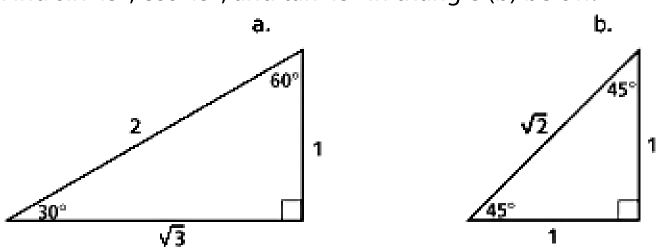
c. Find tan A and tan B.

> **Try It Online!** **www.learner.org**
>
> Problems C3 and C4 can also be explored online as an Interactive Activity. Go to the *Geometry* Web site at www.learner.org/learningmath and find Session 8, Part C.

Problem C5.

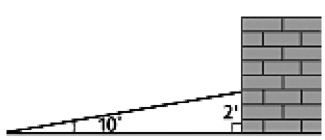
a. Find sin 30º, cos 30º, and tan 30º in triangle (a) below. Then find sin 60º, cos 60º, and tan 60º.

b. Find sin 45º, cos 45º, and tan 45º in triangle (b) below.



Problem C6. Suppose you have any right triangle ABC (with the right angle at C). Explain why it must be true that sin A = cos B.

Problem C7. Most public buildings were built before wheelchair-access ramps became widespread. When it came time to design the ramps, the doors of buildings were already in place. Suppose a particular building has a door two feet off the ground. How long must a ramp be to reach the door if the ramp is to make a 10° angle with the ground? (This is why so many access ramps must make one or more turns!)



sin 10° ≈ 0.17 and cos 10° ≈ 0.98 **[See Note 10]**

Note 10. The values for sine and cosine have been calculated for many triangles. You can find them in a trigonometry table, or you can calculate them yourself using a scientific calculator.

Problem C7 is taken from *Connected Geometry*, developed by Educational Development Center Inc., p. 341. © 2000 Glencoe/McGraw-Hill. Used with permission. www.glencoe.com/sec/math

Homework

Problem H1. Draw any triangle on your paper. Find the three midpoints and connect them in order. Explain why the four triangles created are all similar to the original.

Problem H2. In Problem H1, how do the sides of the new triangles compare to the sides of the original? How do their areas compare?

Problem H3. Draw a square on your paper. Then draw a square with sides 3 times as long. How many times will your original square fit inside the new square?

Problem H4. If two polygons are similar, and one has sides that are r times as long as the sides of the other, how will their areas compare? Explain your answer.

Problem H5. Suppose that a tree's shadow is 21 feet long and a yardstick's shadow is 18 inches long. Using the method from B3, find the height of the tree.

Suggested Reading

This reading is available as a downloadable PDF file on the *Geometry* Web site. Go to www.learner.org/learningmath.

Schifter, Deborah (February 1999). Learning Geometry: Some Insights Drawn from Teacher Writing. *Teaching Children Mathematics, 5,* (5), 360-366. Reproduced with permission from *Teaching Children Mathematics*. ©1999 by the National Council of Teachers of Mathematics.

Tips

Part A: Scale Drawings

Tip A1. Try to come up with a method of comparing the pictures that will tell you whether a picture is distorted. For example, you may want to look at the ratio of length to height. Looking at angles would be another method.

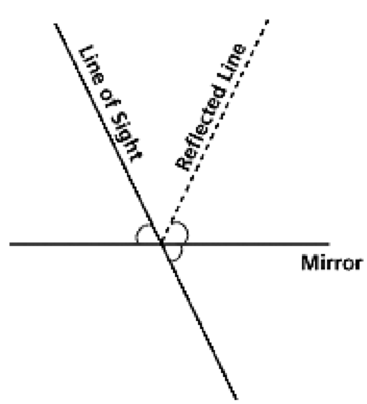
Part B: Similar Triangles

Tip B1. It's important to know how mirrors work to understand this situation: The angle of incidence (the angle at which the light strikes the mirror) is equal to the angle of reflection (the angle at which the light leaves the mirror). **[See Note 5]**

Part C: Trigonometry

Tip C1. Set up a proportion between the corresponding sides of the two triangles, the original one and the new one with unknown sides. There are many different proportions you might set up depending on what you're looking for (e.g., $a_1/c_1 = a_2/c_2$, or $a_1/a_2 = c_1/c_2$, etc.). In each proportion try to use three known side lengths, so the only unknown is the side you're looking for.

Note 5. The mirror acts like there's an exact copy of the world directly behind it that you are peeking into. If you connect your eyes to the top of the other person's head, as if the line went through the mirror, the angles would be the same because they are vertical angles. So the reflected angles are the same as the "behind-the-mirror angles," which are the same as the angles at which you are looking.



Solutions

Part A: Scale Drawings

Problem A1. Picture #3 seems to be the most faithful replica of the original picture because proportions between the various body parts are best preserved.

Problem A2. The two rectangles are similar. Their angles are the same and equal to 90º. To check whether the sides are in proportion, first set up the following ratios of the corresponding sides:

$$\frac{\text{Shorter Side of Smaller Rectangle}}{\text{Longer Side of Smaller Rectangle}} \qquad \frac{\text{Shorter Side of Larger Rectangle}}{\text{Longer Side of Larger Rectangle}}$$

If the two are equal, the sides are in proportion. So we have 2/6 = 3/9, and they are both equal to 1/3. So we can say that the corresponding sides are 1:3 ratio.

Problem A3. The two triangles are similar. By measuring pairs of sides, we can see that they are proportional. In addition, we can measure pairs of corresponding angles and see that they are of equal measure.

Problem A4.

a. The coordinates of the figure are as follows:

A = (0,2)

B = (0.5,3)

C = (1,2)

D = (2,2)

E = (2.5,3)

F = (3,2)

G = (2,0)

H = (1,0)

b. The coordinates of the new figure are as follows:

A' = (0,4)

B' = (1,6)

C' = (2,4)

D' = (4,4)

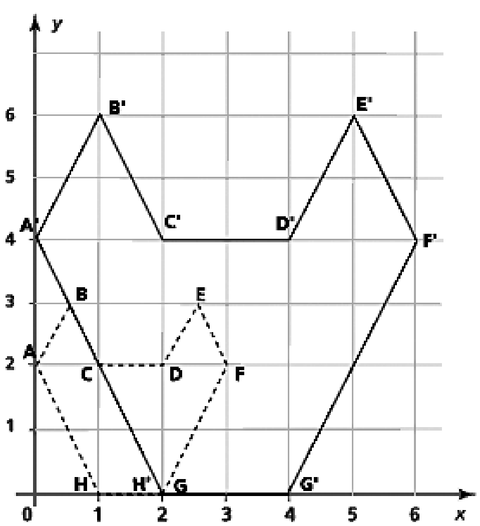
E' = (5,6)

F' = (6,4)

G' = (4,0)

H' = (2,0)

c. Follow the instructions—see figure below.



d. The two figures are similar. We can see this by checking that the corresponding angles have the same measure. The corresponding sides are in a 1:2 proportion by construction.

Solutions, cont'd.

Problem A5. If you changed the angles, you wouldn't have the same kind of shape anymore. For example, in a square, all of the angles are 90°. If you double them all, you will simply have a straight line (a bunch of 180° angles lined up), which is certainly not "similar" to a square. If we start with a triangle, the angle sum is 180°. If we double all of the angles, we would end up with something with angles summed to 360°—certainly not a triangle, and not similar to what we started with.

Problem A6. The distance between the two points is $\sqrt{(a_2 - b_2)^2 + (a_1 - b_1)^2}$.

Problem A7. The new coordinates are $(2b_1,2b_2)$ and $(2a_1,2a_2)$. The new distance is $\sqrt{(2a_2 - 2b_2)^2 + (2a_1 - 2b_1)^2}$. The new distance is double the original distance. We can show this by factoring the 2^2 from the expression, obtaining the following:

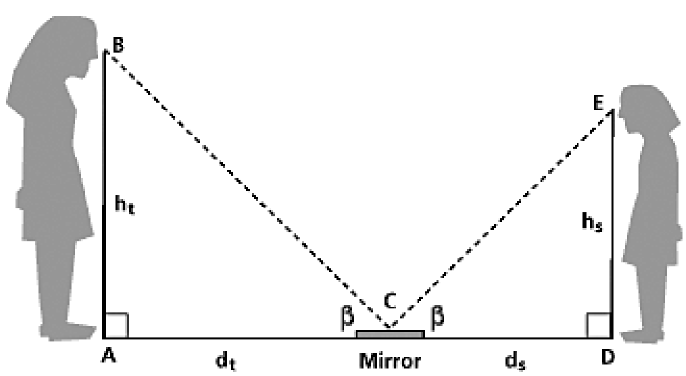
$$\sqrt{4(a_2 - b_2)^2 + 4(a_1 - b_1)^2} = 2\sqrt{(a_2 - b_2)^2 + (a_1 - b_1)^2}.$$

So the new distance is exactly twice the original distance.

Problem A8. We would need to find at least two well-defined reference points on the figure (such as the two endpoints of the spirals). In relation to these two points and the line they define, we could establish pairs of corresponding points on the two shapes. We would then require that all of the distances between all pairs of corresponding points be proportional.

Part B: Similar Triangles

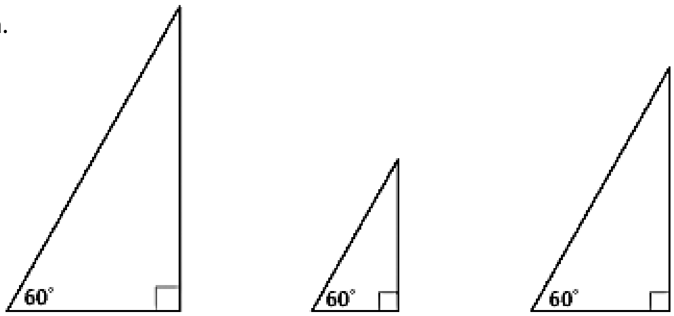
Problem B1. The teacher places the mirror at point *C*, a distance d_s away from the student (see picture). She then steps away from the mirror until she sees the top of the student's head in the mirror. Let's call the distance from the teacher to the mirror d_t. The teacher knows her height, h_t, and she knows that the angle of incidence equals the angle of reflection when a beam of light hits a reflective surface. We call this angle ß. Since the triangles ABC and DEC are right triangles, and since they share the angle ß, they are similar. So the teacher knows that, once she measures d_t and d_s, by similarity of the two triangles, she can say that $h_t/d_t = h_s / d_s$ or $h_s = (h_t \cdot d_s)/d_t$. In other words, by knowing her own height, and by measuring her own as well as the student's distance from the mirror, she can calculate the student's height.



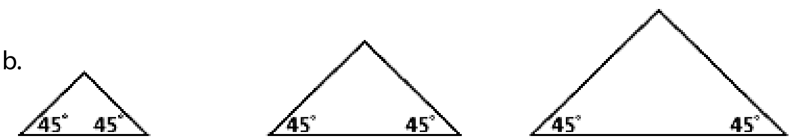
Solutions, cont'd.

Problem B2. We can conjecture that two triangles are similar if two of their respective angles have the same measure.

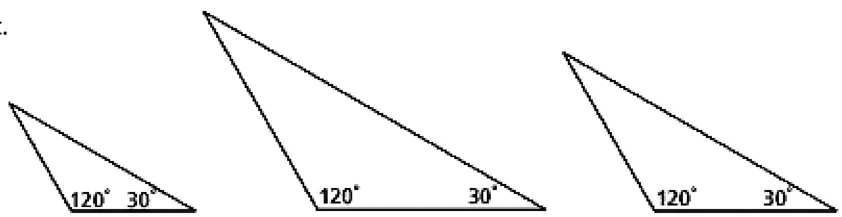
a.



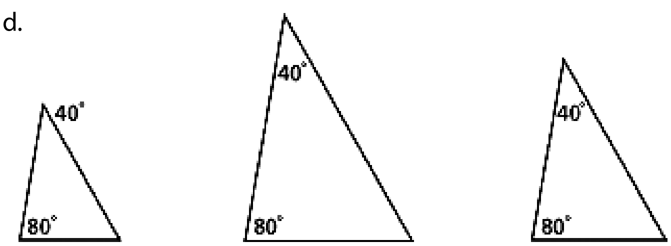
b.



c.



d.



Problem B3.

a. Nancy is taller since the right triangles defined by their heights and their shadows are similar, so the bases of the triangles have to be proportional to the heights of the triangles (i.e., their body heights).

b. Converting Michelle's height into inches (64 inches) and setting up a proportion, you would have:

$64 / x = 96 / 102$, or

$x = 68$

Converting 68 inches back to feet, Nancy is 5 feet 8 inches tall.

c. Converting Nancy's height into inches (64 inches) and setting up a proportion, you would have:

$64 / x = 102 / 96$, or

$x = 60.24$

Converting 60.24 inches back to feet, Michelle is approximately 5 feet and 1/4 inch tall.

Solutions, cont'd.

Part C: Trigonometry

Problem C1. The triangle with hypotenuse 4: By similarity with the original triangle, we can say:

$5/4 = 3/a$, so $a = 12/5$.

The triangle with hypotenuse 10: By similarity with the original triangle, we can say:

$5/10 = 3/a$, so $a = 30/5 = 6$.

The triangle with hypotenuse 1: By similarity with the original triangle, we can say:

$5/1 = 3/a$, so $a = 3/5$.

Problem C2. The triangle with hypotenuse 4: By similarity with the original triangle, we can say:

$5/4 = 4/b$, so $b = 16/5$.

The triangle with hypotenuse 10: By similarity with the original triangle, we can say:

$5/10 = 4/b$, so $b = 40/5 = 8$.

The triangle with hypotenuse 1: By similarity with the original triangle, we can say:

$5/1 = 4/b$, so $b = 4/5$.

Problem C3.

a. sin A = 3/5 and cos A = 4/5
b. sin B = 4/5 and cos B = 3/5

Problem C4. We calculate the hypotenuse with the Pythagorean theorem and find that it's 13.

a. sin A = 12/13, cos A = 5/13
b. sin B = 5/13, cos B = 12/13
c. tan A = 12/5, tan B = 5/12

Problem C5.

a. $\sin 30^\circ = \cos 60^\circ = 1/2$
   $\sin 60^\circ = \cos 30^\circ = \sqrt{3} / 2$
   $\tan 30^\circ = 1/\sqrt{3}$
   $\tan 60^\circ = \sqrt{3}$

These values are constant for any triangle with angles 30º-60º-90º.

b. $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$
   $\tan 45^\circ = 1$

These values are constant for any triangle with angles 45º-45º-90º.

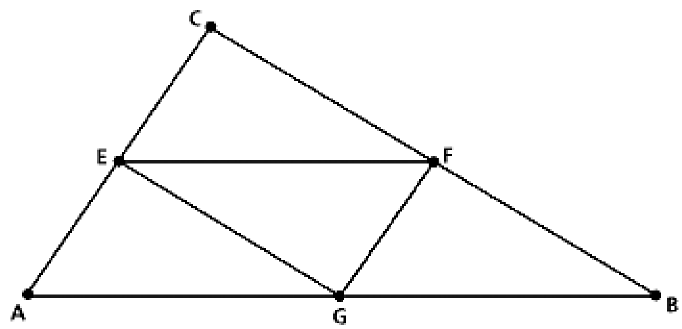
Problem C6. Angles A and B are adjacent, so sin A = cos B. The hypotenuse is the same for both angles, but the roles of "adjacent side" and "opposite side" switch. The side opposite angle B is adjacent to angle A, and vice versa.

Solutions, cont'd.

Problem C7. Let x be the length of the ramp. Then we have a right triangle with hypotenuse x, shorter leg 2, and the angle opposite to the shorter leg of 10°. Since $\sin 10° = 2/x$, we have $x = 2/\sin 10° = 2/0.17 \approx 11.765$ feet.

Homework

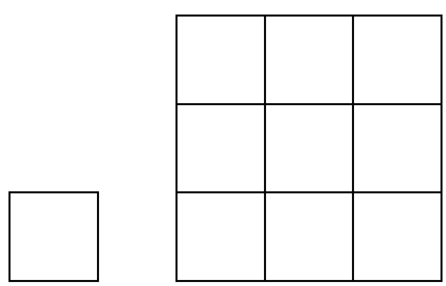
Problem H1.



The midline theorem in Session 5 tells us that the segment connecting two midpoints is half as long as the third side of a triangle. So for all four triangles, we can see that the sides are half as long as the sides of the original triangle, so they are similar to it by SSS.

Problem H2. The sides of the new triangles are half the length of the corresponding sides in the original triangle. The areas of the new triangles are one-fourth of the area of the original triangle.

Problem H3. Nine times.



Problem H4. The area of the polygon whose sides are r times as long will have an area that is r^2 times the area of the original polygon. Proving this assertion involves recalling that any polygon can be divided into triangles. By the way the two polygons relate, the sides of all of the triangles inside one of the polygons will be r times longer than the sides of the triangles inside the other polygon. By Problem H3, each constituent triangle in the polygon whose sides are r times longer will have an area r^2 times greater. So the entire area will be r^2 times greater.

Problem H5. We convert all measures into feet and set up a proportion based on the similarity of right triangles whose heights are the yardstick and the tree and whose bases are the respective shadows. So we get that, if x is the height of the tree, then:

$x/3 = 21/1.5$

It follows that $x = 42$, so the tree is 42 feet tall.