Session 7

Symmetry

Key Terms for This Session

Previously Introduced

- coordinates
- reflection
- rotation
- translation

New in This Session:

- frieze pattern
- glide reflection
- reflection or line symmetry
- rotation symmetry
- symmetry
- translation symmetry
- vector

Introduction

Symmetry is one of the most important ideas in mathematics. There can be symmetry in an algebraic calculation, in a proof, or in a geometric design. It's such a powerful idea that when it's used in solving a problem, we say that we exploit the symmetry of the situation. In this session, you will explore geometric versions of symmetry by creating designs and examining their properties.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will do the following:

- Learn about geometric symmetry
- Explore line or reflection symmetry
- Explore rotation symmetry
- Explore translation symmetry and frieze patterns

Note 1.

Materials Needed:

- Mira (a transparent image reflector)
- tracing paper or patty paper and fasteners
- large paper or poster board (optional)
- scissors
- markers

Mira
You can purchase a Mira from the following source:

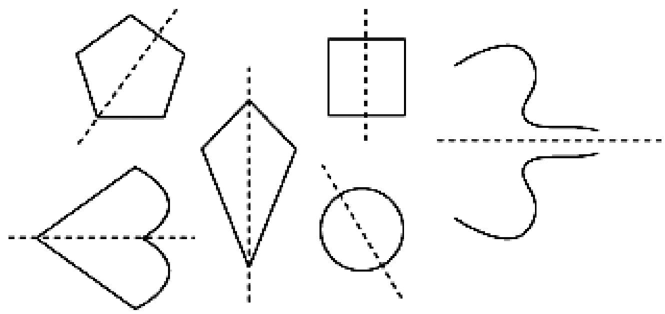
ETA/Cuisenaire
500 Greenview Court
Vernon Hills, IL 60061
800-445-5985/847-816-5050 • 800-875-9643/847-816-5066 (fax) • http://www.etacuisenaire.com/

Part A: Line Symmetry (30 minutes)

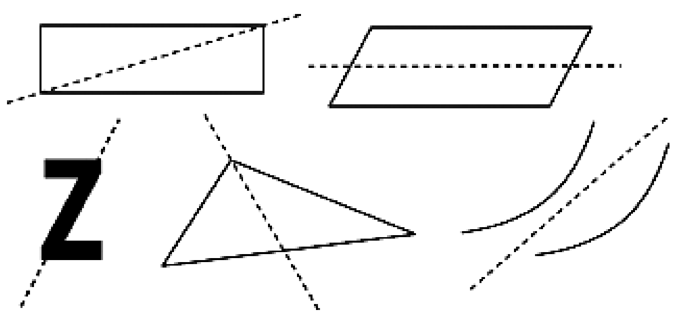
Finding Lines of Symmetry

In Session 5, you saw three ways to move figures around: rotation, reflection, and translation. If you can move an entire design in one of these ways, and that design appears unchanged, then the design is symmetric.

If you can reflect (or flip) a figure over a line and the figure appears unchanged, then the figure has reflection symmetry or line symmetry. The line that you reflect over is called the line of symmetry. A line of symmetry divides a figure into two mirror-image halves. The dashed lines below are lines of symmetry:



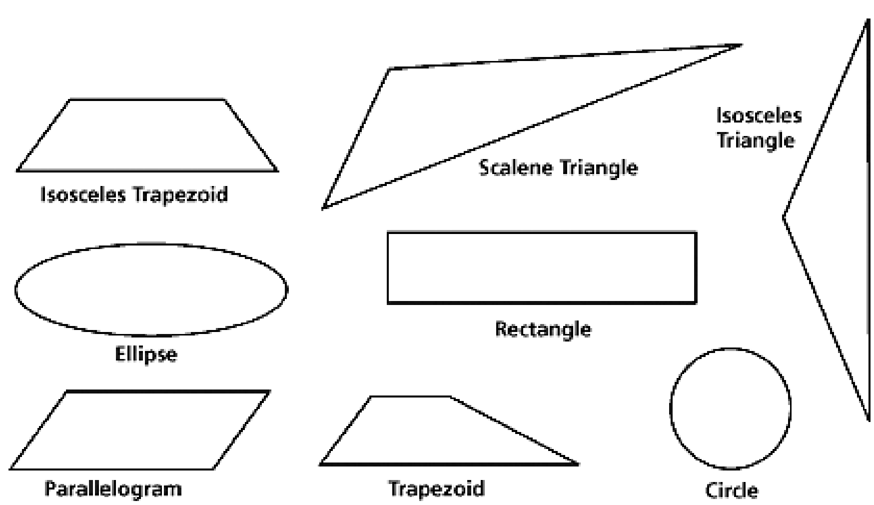
The dashed lines below are not lines of symmetry. Though they do cut the figures in half, they don't create mirror-image halves.



You can use a Mira (image reflector) or simply the process of cutting and folding to find lines of symmetry. Take out page 166, a full-size page of the figures shown here, and compare the lines proposed using a Mira.

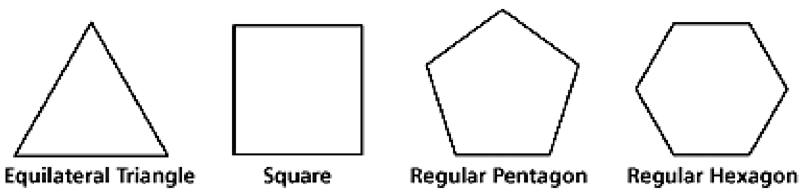
In Problems A1 and A2, show the lines of symmetry as dashed lines on the figures below.

Problem A1. For each figure, find all the lines of symmetry you can.



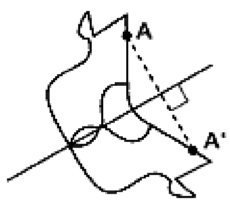
Part A, cont'd.

Problem A2. Find all the lines of symmetry for these regular polygons. Generalize a rule about the number of lines of symmetry for regular polygons.

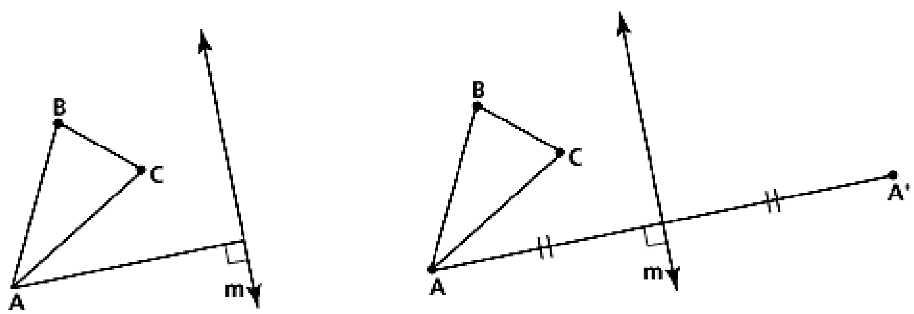


The Perpendicular Bisector

If point A′ is the mirror image of point A in a figure with a line of symmetry, then the line of symmetry is the perpendicular bisector of the segment AA′.



You can use that fact to reflect figures over lines without using a device like a Mira. Suppose you want to reflect the triangle below over the line shown. To reflect point A, draw a segment from A perpendicular to the line. Continue the segment past the line until you've doubled its length. You now have point A′, the mirror image of A.



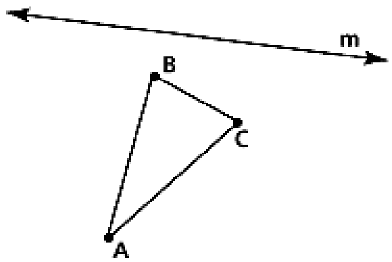
Repeat the process for the other two endpoints, and connect them to form the triangle.

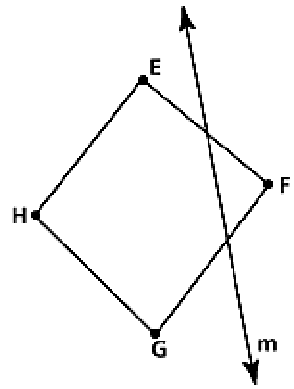
Try It Online!	**www.learner.org**
This problem can be explored online as an Interactive Illustration. Go to the Geometry Web site at www.learner.org/learningmath and find Session 7, Part A.	

Part A: Finding Lines of Symmetry is adapted from *IMPACT Mathematics, Course 3*, developed by Educational Development Center, Inc., pp. 289-290. www.glencoe.com/sec/math

Part A, cont'd.

Problem A3. For each figure, reflect the figure over the line shown using perpendicular bisectors. Check your work with a Mira. The figures can be found on pages 167 and 168.





Video Segment (approximate time: 11:09-12:45): You can find this segment on the session video approximately 11 minutes and 9 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, the participants explain different ways in which they came up with perpendicular bisectors to reflect the figures over the line of symmetry.

Did you use the same method as shown in the video segment? Did you come up with a different way of solving the problems?

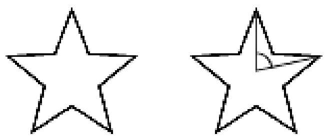
Problem A3 and the preceding text are adapted from *IMPACT Mathematics, Course 3*, developed by Educational Development Center, Inc., pp. 295-296. www.glencoe.com/sec/math

Part B: Rotation Symmetry (30 minutes)

Determining Rotation Symmetry

If you can rotate (or turn) a figure around a center point by fewer than 360° and the figure appears unchanged, then the figure has rotation symmetry. The point around which you rotate is called the center of rotation, and the smallest angle you need to turn is called the angle of rotation.

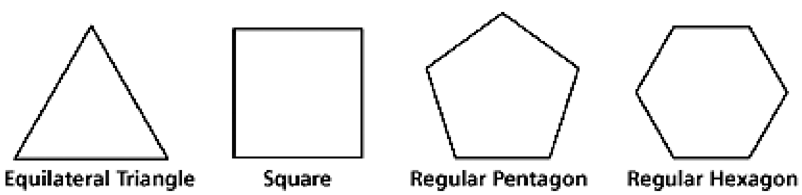
This figure has rotation symmetry of 72°, and the center of rotation is the center of the figure:



Problem B1.

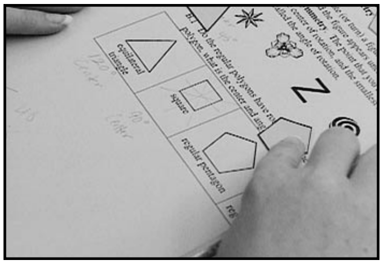
a. Each of these figures has rotation symmetry. Can you estimate the center of rotation and the angle of rotation?

b. Do the regular polygons have rotation symmetry? For each polygon, what are the center and angle of rotation?



Try It Online!	**www.learner.org**
This problem can be explored online as an Interactive Activity. Go to the Geometry Web site at www.learner.org/learningmath and find Session 7, Part B.	

Part B, cont'd.



Video Segments (approximate times: 16:49-17:22 and 18:30-20:23): You can find the first segment approximately 16 minutes and 49 seconds after the Annenberg/CPB logo. The second segment begins approximately 18 minutes and 30 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, watch the participants as they explore rotational symmetry and try to come up with the rule for regular polygons' rotational symmetry.

Were you able to come up with the rule? Does the rule work only for regular polygons or also for irregular ones? **[See Note 2]**

As you will see in the next section, in order to have rotation symmetry, the center of rotation does not have to be the center of the figure. A figure can have rotation symmetry about a point that lies outside the figure.

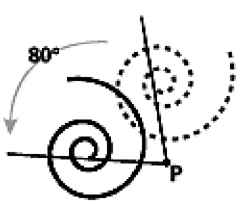
Creating Rotation Symmetry

To create a design with rotation symmetry, you need three things:

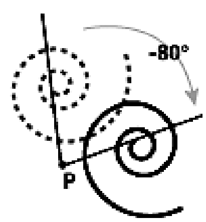
- a figure, called a basic design element, that you will rotate
- a center of rotation
- an angle of rotation

By convention, we rotate figures counterclockwise for positive angles and clockwise for negative angles.

80° rotation about point P

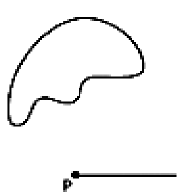


-80° rotation about point P



To create a symmetric design, follow the steps below:

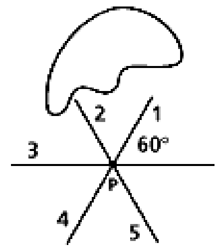
Step 1: Copy this picture, including point P and the reference line. Or take out page 169 of the design.



Note 2. Note that the angle of rotation, which is equal to the external angle, is also equal to the central angle of the polygon. For regular polygons, the central angle has its vertex at the center of the polygon, and its rays go through any two adjacent vertices. In Session 4, we defined central angles in terms of circles. Here, you can think of circumscribing a circle about the regular polygon. Then these two notions of central angle coincide.

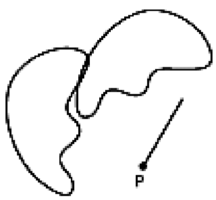
Part B, cont'd.

Step 2: Draw a new segment with P as one endpoint and forming a 60° angle with the reference line. Label this segment 1. Then draw four more segments, labeling them 2 through 5, from point P, each forming a 60° angle with the previous segment, as shown here:



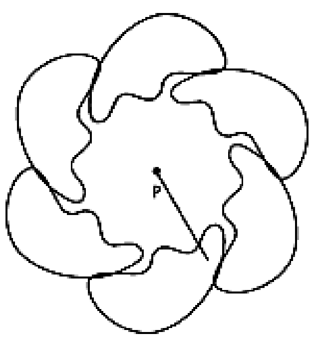
Step 3: Place a sheet of tracing paper over your figure. Pin the papers together through the center of rotation. Trace the figure, including the reference line, but don't trace segments 1-5.

Step 4: Now rotate your tracing until the reference line on the tracing is directly over segment 1. Trace the original figure again. Your tracing should now look like this.



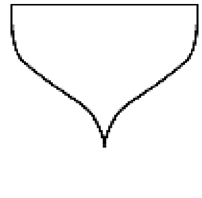
Step 5: Rotate the tracing until the reference line on the tracing is directly over segment 2. Trace the original figure again.

Step 6: Repeat the process, rotating to place the reference line over the next segment and tracing the figure. Do this until the reference line on the tracing is back on the original reference line.



Problem B2. Does your design have reflection symmetry? If so, where is the line of symmetry?

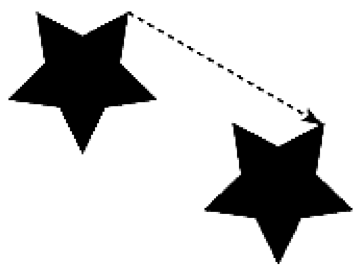
Problem B3. Use the basic design element at right and the given center of rotation to create a symmetric design with an angle of rotation of 120°. Does this design have reflection symmetry? If so, where is the line of symmetry? Take out page 170 of the design.



Part C: Translation Symmetry and Frieze Patterns (60 minutes)

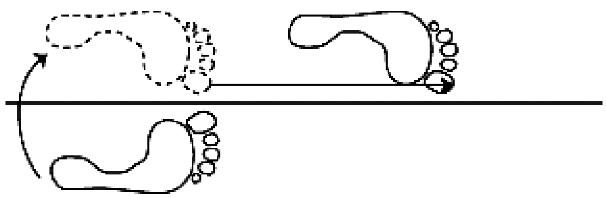
Translation Symmetry

A translation or slide involves moving a figure in a specific direction for a specific distance. Vectors are often used to denote translation, because the vector communicates both a distance (its length) and a direction (the way it is pointing).



The vector shows both the length and direction of the translation.

A glide reflection is a combination of two transformations: a reflection over a line, followed by a translation in the same direction as the line.



Reflect over the line shown; then translate parallel to that line.

Only an infinite strip can have translation symmetry or glide reflection symmetry. For translation symmetry, you can slide the whole strip some distance, and the pattern will land back on itself. For glide reflection symmetry, you can reflect the pattern over some line, then slide in the direction of that line, and it looks unchanged.

The patterns must go on forever in both directions.

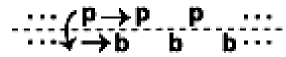
Part C, cont'd.

Frieze Patterns

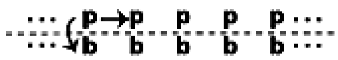
An infinite strip with a symmetric pattern is called a frieze pattern. There are only seven possible frieze patterns if we are using only one color. **[See Note 3]**

1. Translation symmetry only:

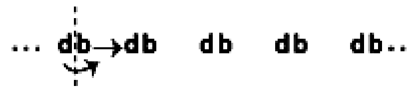
2. Glide reflection plus translation symmetry:



3. Reflection over a horizontal line plus translation:

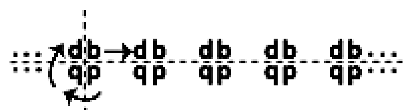


4. Reflection over a vertical line plus translation:



5. Rotation (a half-turn about a point on the midline of the strip) plus translation:

6. Reflection over a vertical line plus reflection over a horizontal line plus translation:



7. Reflection over a vertical line plus glide reflection plus translation:

Problem C1. It may not be obvious how an infinite frieze pattern can be created from a basic element. Follow these step-by-step instructions to create a frieze using Design 6 (reflection over a vertical line plus reflection over a horizontal line plus translation). The instructions use the letter p as a basic design element of the letter. The design element page (page 171) contains several versions of a more complex design element. Take out this page and cut out the design elements. Then create the frieze pattern using this design element. Alternatively, you can draw your own design element to create the frieze pattern.

Step 1: Start with a basic design element. It's best if it is a nonsymmetric design so that the symmetry created by the transformations is more apparent.

Step 2: All frieze patterns have translation symmetry, so we'll leave that for last. Once we create a basic unit that contains all of our required other symmetries, we can translate it infinitely in both directions. So in this case, we'll start with a vertical reflection. Take your basic design element and reflect it over a vertical line. It's best to choose a line that is close to, but not intersecting, your original element.

There are now two pieces to your basic design: the original element and its reflected image.

Note 3. If you are working with a group, consider going through these patterns one by one on a large poster. Have participants come up and show the lines and points in question. As you introduce the seven frieze patterns, work through Problem C1 as a whole group.

Part C, cont'd.

Problem C1, cont'd.

Step 3: The next symmetry is horizontal reflection, so take your basic design block (now two elements) and reflect them both over a horizontal line. Again, choose a line that is close to, but not intersecting, your original design.

```
     db
-----------
     qp
```

Step 4: We now have a basic element with all of the required symmetries except for translation. Take your basic element and translate it by a fixed distance in both directions. You have created a frieze pattern!

```
... db   db   db   db   db ...
... qp   qp   qp   qp   qp ...
```

The seven patterns given are certainly not all the possible combinations of transformations. How can they be the only possible frieze patterns? It turns out that other combinations fall into one of these categories as well. That is, they create equivalent patterns.

Problem C2. Use the design element page (page 171) or draw your own design element to create the seven frieze patterns.

Problem C3. Frieze patterns appear in the artwork of Native American and African cultures, as well as in cornices on buildings. Create a more interesting basic design element, and create a frieze pattern with that element. Choose one of the seven patterns described above.

Classifying Frieze Patterns

Note that the point of the following problems is not to memorize the classification system, but rather to be able to make sense of this kind of a system, interpret what the symbols mean, and apply it to new situations.

Mathematicians have developed a two-character notation to denote each possible frieze pattern.

- The first character is m or 1 according to whether there is reflection over a vertical line or not.
- The second character is m if there is a reflection over a horizontal line, g if there is a glide reflection, 2 if there is a rotation, and 1 if none of these exists.

Take It Further

Problem C4. Classify each of the previous seven frieze patterns according to this system.

Take It Further

Problem C5. The symbol m2 was not part of the classification you used. (It was the only combination of the two symbols missing.)

a. What would m2 represent in terms of symmetries?

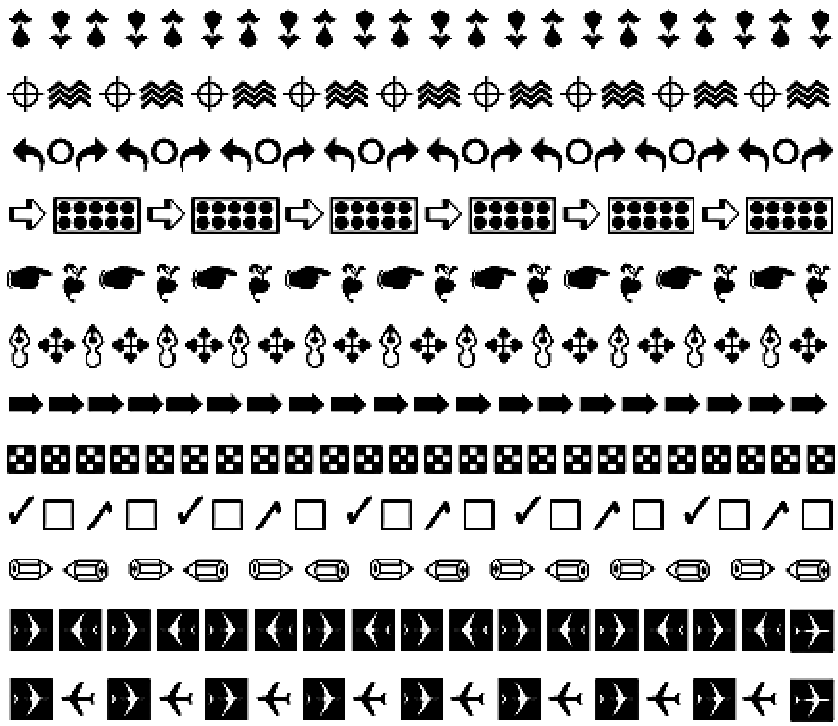
b. Create a simple frieze pattern with m2 symmetry.

c. What is another name for the symmetry in your pattern?

Part C, cont'd.

Take It Further

Problem C6. Using the codes for frieze patterns, classify each of the following designs. Assume the designs are really infinite.



Homework

The coordinate geometry that you first worked with in Session 6 is useful for describing transformations. Before looking at how that works, here are a few problems with coordinates to get you warmed up.

Problem H1. Do each part on a new set of axes.

a. Find three different points with x-coordinate 0. Where are all the points with x-coordinate 0?

b. On a new set of axes, plot the point (1,1). Then draw a horizontal line through that point. Name three other points on that line. What do all the points on that line have in common?

c. Suppose *v* is the vertical line passing the point (3,7). Find the coordinates of three points that are on the line *v* and three points that are not on the line *v*. How can you tell if a point is on line *v* just by looking at the coordinates?

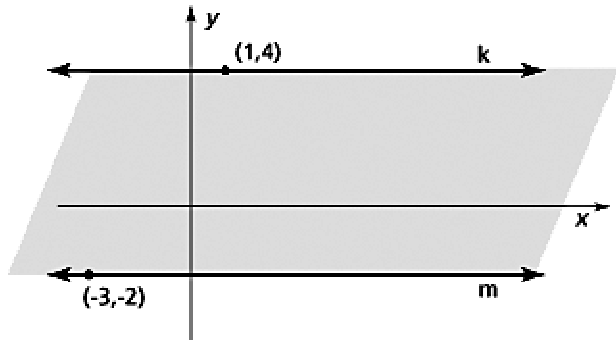
d. What are the coordinates of the intersection of the horizontal line through (5,2) and the vertical line through (-4,3)?

e. Name and plot five points whose first coordinate is the same as the second coordinate. Where are all such points?

Problems in Part C are adapted from the NCTM Addenda Series, developed by North Central Regional Education Laboratory, pp. 34-35 and 42.

Homework, cont'd.

Problem H2. In this picture, *k* and *m* are horizontal lines.



a. Find the coordinates of six points between the lines *k* and *m*.
b. Find the coordinates of six points that are not between the lines *k* and *m*.
c. How can you tell if a point is between the lines *k* and *m* by looking at its coordinates?

Problem H3. Copy and complete the table below:

A	B	C	D	E	F	G
(x,y)	(x + 3,y - 2)	(-x,y)	(2x,2y)	(x - 1,y + 2)	(y,-x)	
(2,1)					(1,-2)	
(-4,0)	(-1,-2)					
(-5,4)						(-4,-5)

a. On a piece of graph paper, plot the three points in column A. Connect them to form a triangle. Plot the three points in column B. Connect them to form a triangle. Describe how the two triangles are related.
b. On a new piece of graph paper, plot triangles A and C. Describe how they're related.
c. Repeat the process for triangles A and D.
d. Repeat the same for the rest of the table.

Homework, cont'd.

Problem H4. Start with any point (x,y). Reflect that point over the x-axis. What are the coordinates of the new point? How do they relate to the coordinates of the original point? Explain. (You may want to try several cases.)

Problem H5. Start with any point (x,y). Reflect that point over the y-axis. What are the coordinates of the new point? How do they relate to the coordinates of the original point? Explain. (You may want to try several cases.)

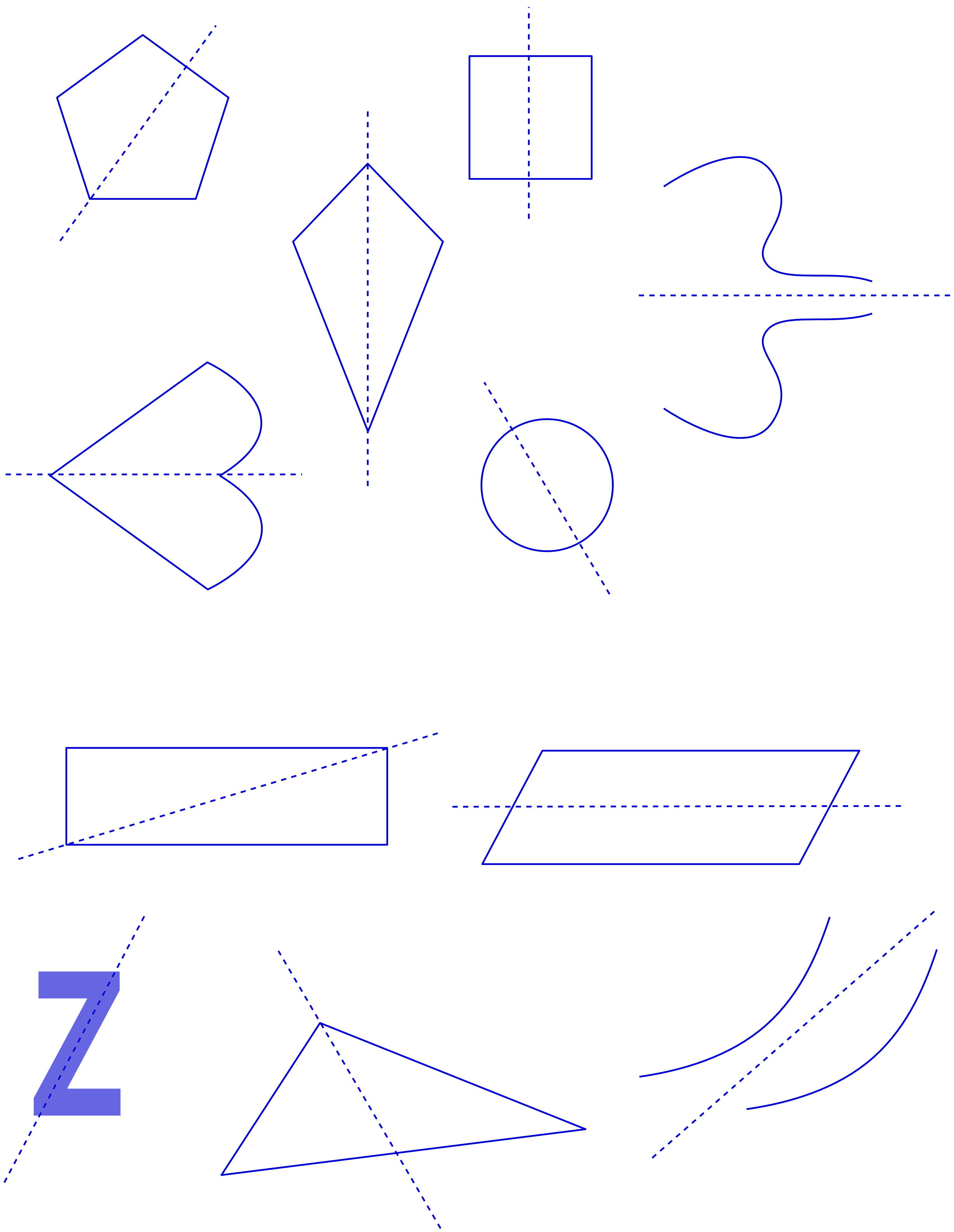
Problem H6. Start with any point (x,y). Reflect that point over the line y = x. What are the coordinates of the new point? How do they relate to the coordinates of the original point? Explain. (You may want to try several cases.)

Suggested Reading

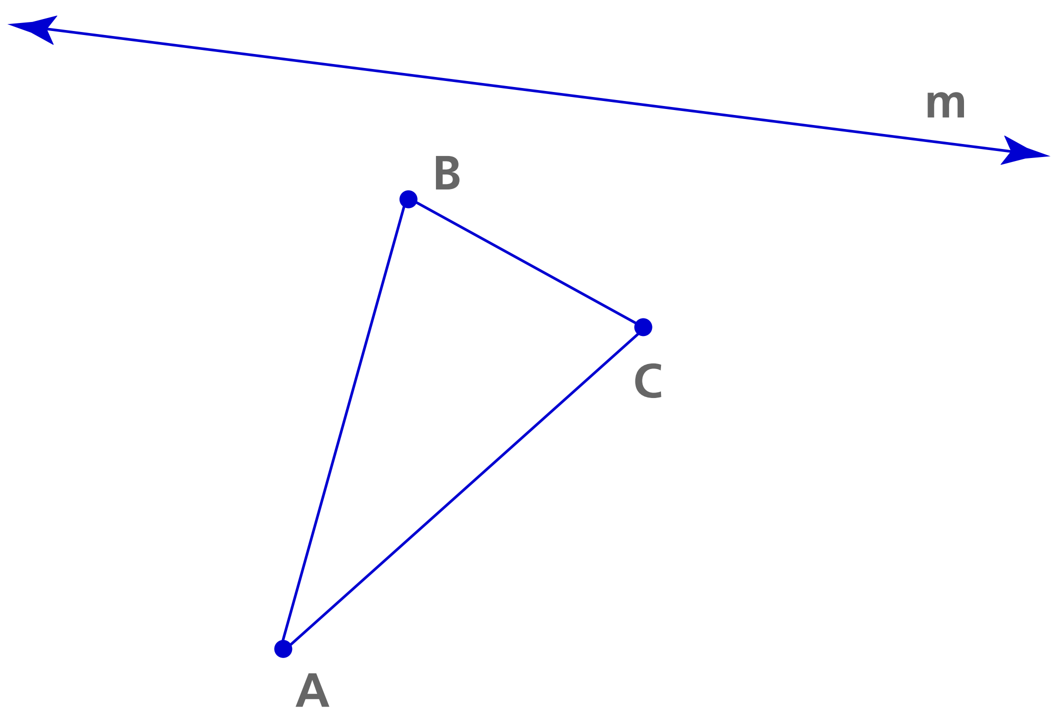
This reading is available as a downloadable PDF file on the *Geometry* Web site. Go to www.learner.org/learningmath.

Crowe, D., and Thompson, Thomas M. (1987). Transformation Geometry and Archaeology. In *Learning and Teaching Geometry K-12* (pp. 106-109). © 1987 by the National Council of Teachers of Mathematics. All rights reserved.

Figures for Part A,
Finding Lines of Symmetry



Figures for Problem A3



Figures for Problem A3, cont'd.

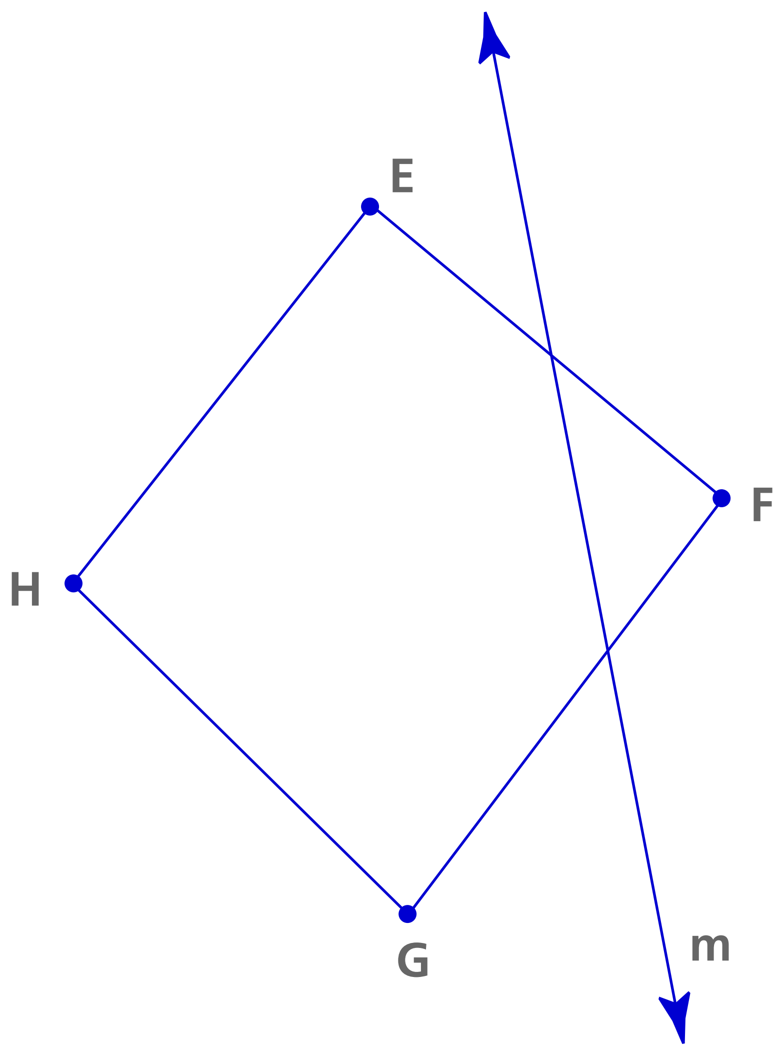


Figure for Part B, Creating Rotation Symmetry

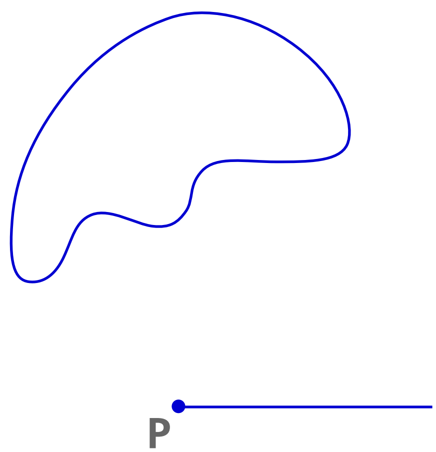
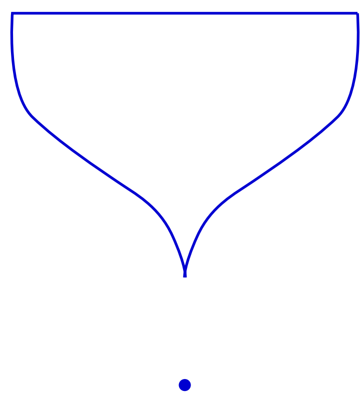
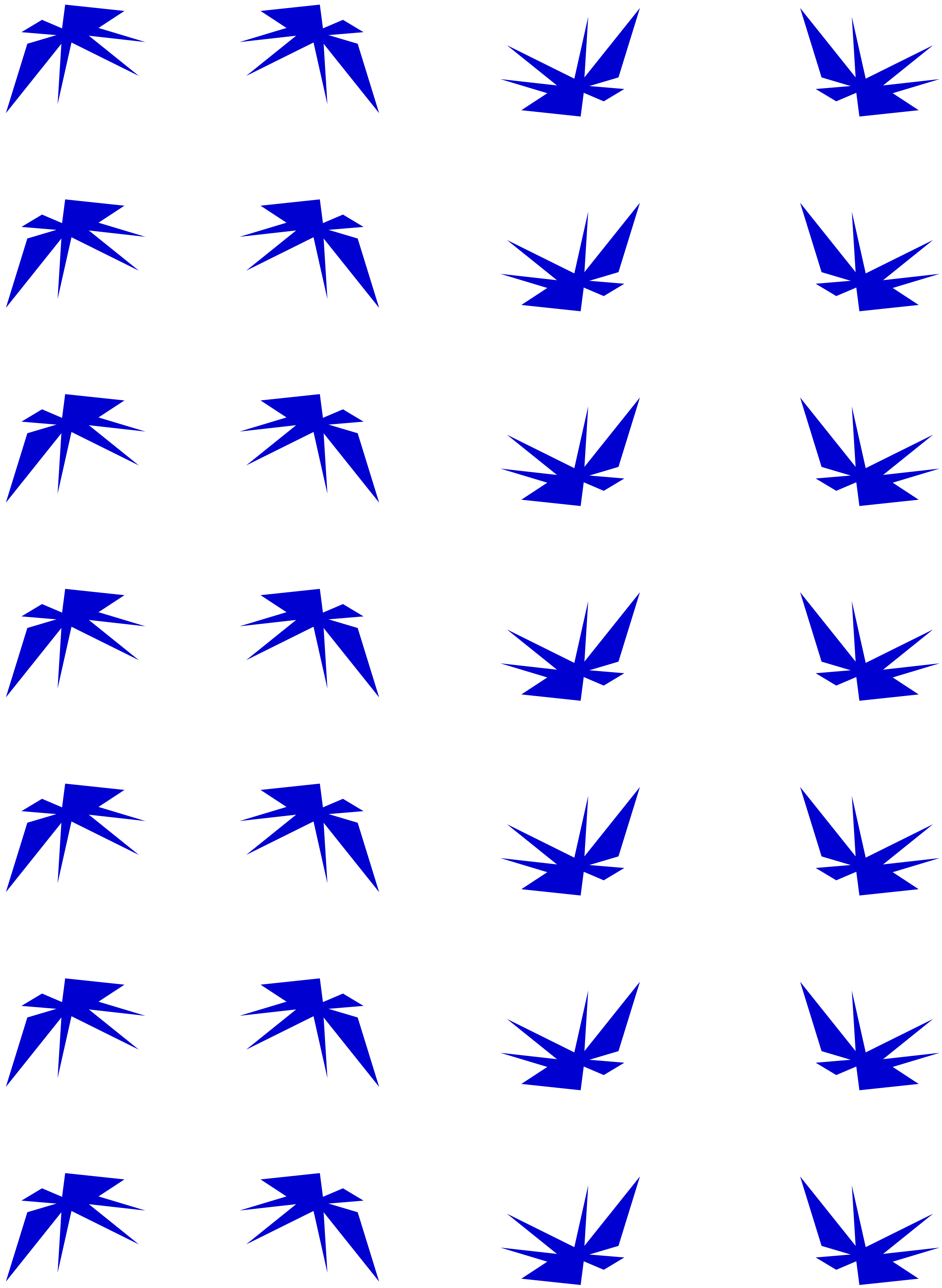


Figure for Problem B3



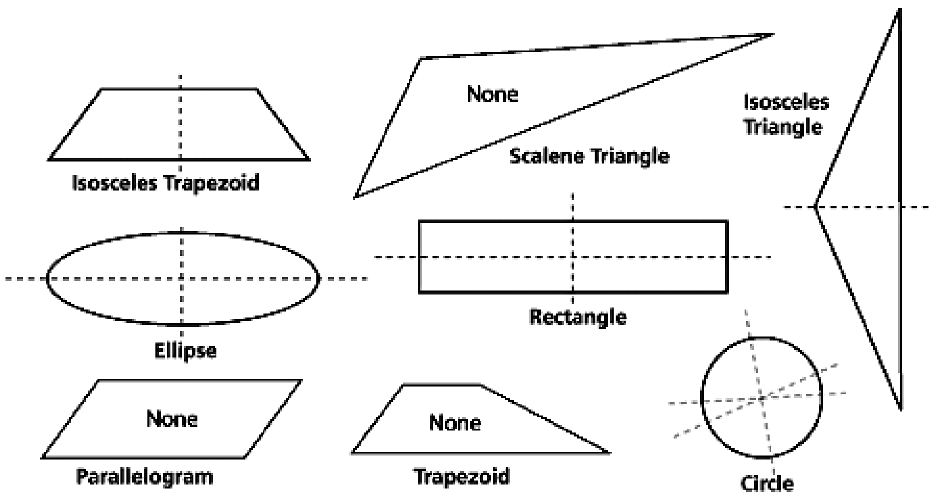
Figures for Problem C1



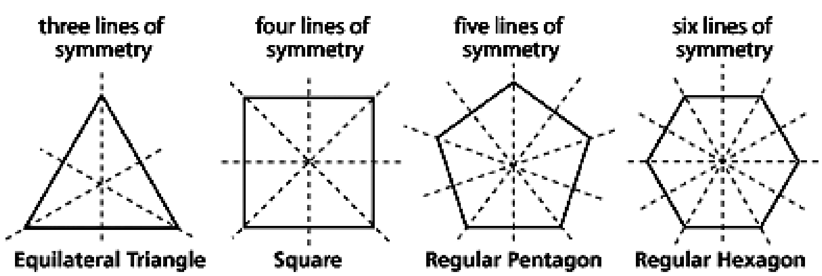
Solutions

Part A: Line Symmetry

Problem A1. The isosceles trapezoid has one line of symmetry, the perpendicular bisector of the base. The scalene triangle has no lines of symmetry. The isosceles triangle has one line of symmetry, the perpendicular bisector of the base. The ellipse has two lines of symmetry, one along the major and one along the minor axis. The rectangle has two lines of symmetry, the perpendicular bisector of the longer sides and the perpendicular bisector of the shorter sides. The circle has infinitely many lines of symmetry, any line going through the center. (Any diameter is a line of symmetry.) The parallelogram pictured has no lines of symmetry. Neither does the trapezoid.

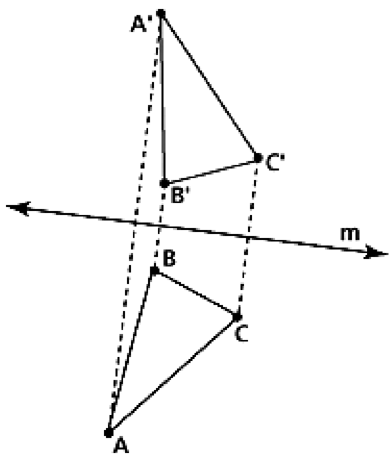


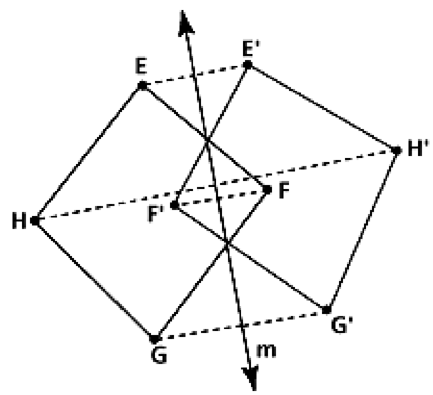
Problem A2. Each regular polygon has as many lines of symmetry as it has sides.



Solutions, cont'd.

Problem A3.





Part B: Rotation Symmetry

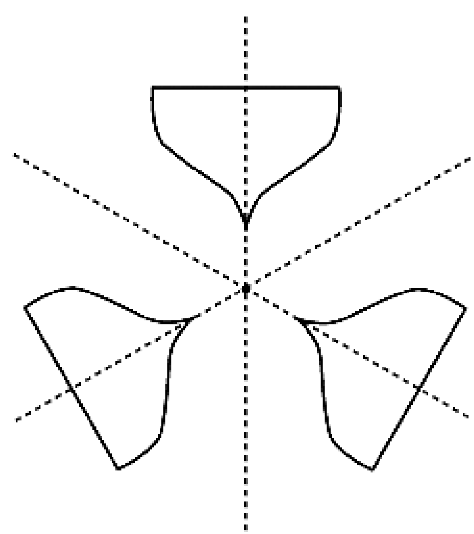
Problem B1.

a. For each shape, the center of rotation is the center of the figure. The angles of rotation, from left to right, are 120°, 180°, 120°, and 90°.

b. Regular polygons do have rotation symmetry. In each case, the center of rotation is the center of the polygon, and the angle of rotation is 360°/*n*, where *n* is the number of sides in the polygon. So the angles are 120°, 90°, 72°, and 60°, respectively, for the equilateral triangle, square, regular pentagon, and regular hexagon.

Problem B2. The design has no reflection symmetry.

Problem B3. The final design has three lines of symmetry. Each line of symmetry goes through the center of the rotation symmetry and is a perpendicular bisector of the straight line on the basic design element. Notice that the line symmetry arose because of line symmetry in the original basic design element, something that the first picture lacked.



Solutions, cont'd.

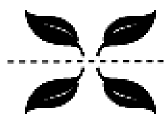
Part C: Translation Symmetry and Frieze Patterns

Problem C1.

Step 1:

Step 2:

Step 3:



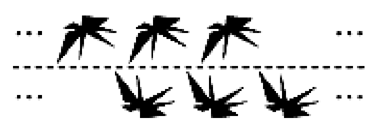
Step 4:

Problem C2.

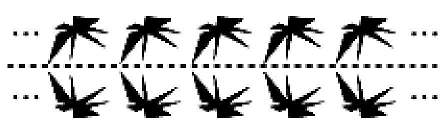
Using a new design element, the seven patterns are as follows:

1. Translation symmetry only:

2. Glide reflection plus translation symmetry:

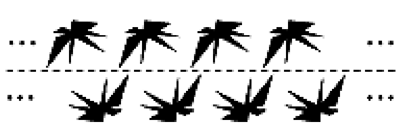


3. Reflection over a horizontal line plus translation:

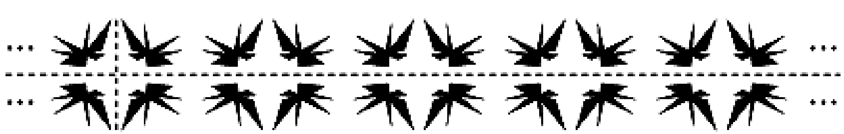


4. Reflection over a vertical line plus translation:

5. Rotation (a half-turn about a point on the midline of the strip) plus translation:



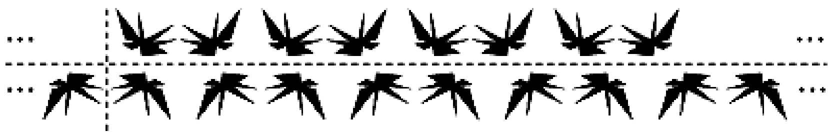
6. Reflection over a vertical line plus reflection over a horizontal line plus translation:



Solutions, cont'd.

Problem C2, cont'd.

7. Reflection over a vertical line plus glide reflection plus translation:



Problem C3. Answers will vary depending on the choice of the basic element as well as of the pattern.

Problem C4. Following the order in which the patterns are originally described, they can be classified as 11, 1g, 1m, m1, 12, mm, mg.

Problem C5.

a. It would be a combination of translation symmetry (which is present in all patterns), followed by a reflection about a vertical line (hence the m), followed by the 180° rotation about a point on the midline (hence the 2).

b. See picture:

```
...   pq  pq  pq  pq   ...
... bd  bd  bd  bd  bd ...
```

c. The symmetry described is equivalent to pattern #7, so it can also be described as having translation symmetry, glide reflection, and reflection over a vertical line. So it could also be classified as mg.

Problem C6. If you consider symmetry in the basic design element part of the symmetry of the frieze pattern, you may have slightly different answers from the ones here. In the order in which the designs are given, the classifications are as follows: 12, 11 (or possibly m1), m1, 11 (or possibly 1m), 11, 11 (or possibly m1), 11 (or possibly 1m), 11 (or possibly mm), 12, m1 (or possibly mm), m1 (or possibly mm), and 11 (or possibly 1m).

Homework

Problem H1.

a. Any three points on the vertical axis will do—for instance, (0,-3), (0,1), and (0,13). All the points with x-coordinate 0 are on the y-axis.

b. Any three points with the y-coordinate of 1 will do—for instance, (-7,1), (0,1), and (12,1). All the points on this horizontal line have y-coordinate 1.

c. Any three points with x-coordinate 3 will do—for instance, (3,-4), (3,0), and (3,11). A point is on the line *v* if its x-coordinate is 3. If its x-coordinate is anything other than 3, the point is not on the line.

d. The coordinates are (-4,2).

e. For instance, (-3,-3), (-2,-2), (0,0), (4,4), (15,15). All of these points are on the line y = x.

Solutions, cont'd.

Problem H2.

a. There are infinitely many points between the two lines—for instance, (-32,-1), (-17,1) (0,0), (33,3), (155,3.5), (1000,3.9).

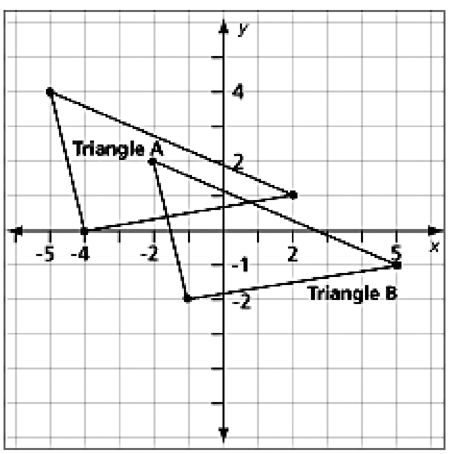
b. There are infinitely many points which are not between the two lines—for instance, (-32,7), (-17,-15), (0,5), (33,7), (155,4.5), (1000,7.9).

c. A point is between the two lines if its y-coordinate is greater than -2 and less than 4.

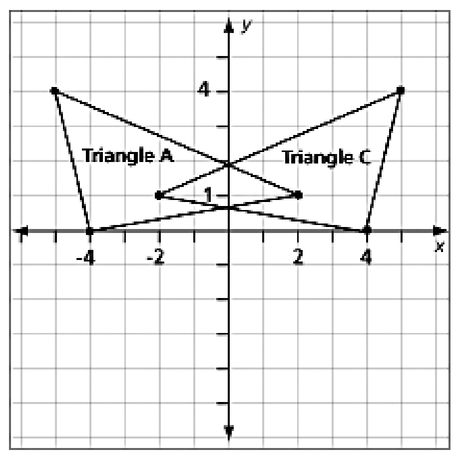
Problem H3.

A	B	C	D	E	F	G
(x,y)	(x + 3,y - 2)	(-x,y)	(2x,2y)	(x - 1,y + 2)	(y,-x)	(-y,x)
(2,1)	(5,-1)	(-2,1)	(4,2)	(1,3)	(1,-2)	(-1,2)
(-4,0)	(-1,-2)	(4,0)	(-8,0)	(-5,2)	(0,4)	(0,-4)
(-5,4)	(-2,2)	(5,4)	(-10,8)	(-6,6)	(4,5)	(-4,-5)

a. Triangle B is the translation of triangle A 3 units to the right and 2 units down:



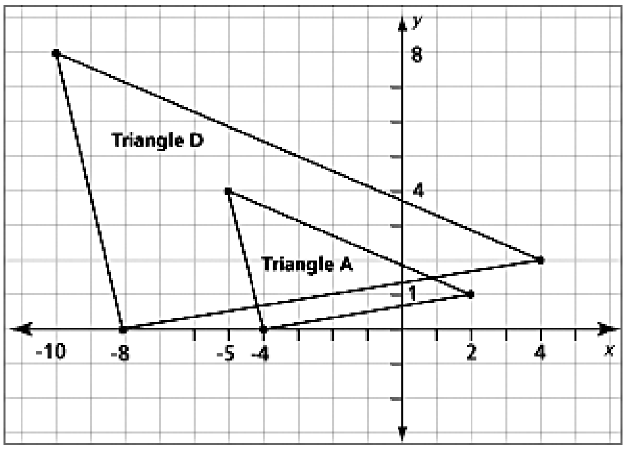
b. Triangle C is obtained by reflecting triangle A about the vertical axis:



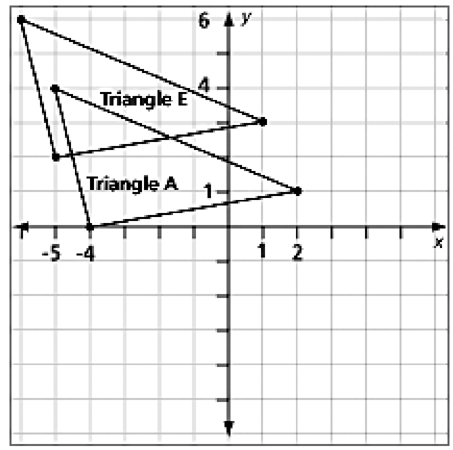
Solutions, cont'd.

Problem H3, cont'd.

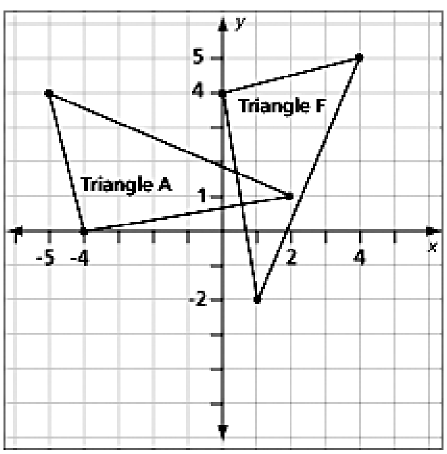
c. Triangle D is obtained by stretching triangle A in both x- and y-direction by a factor of 2:



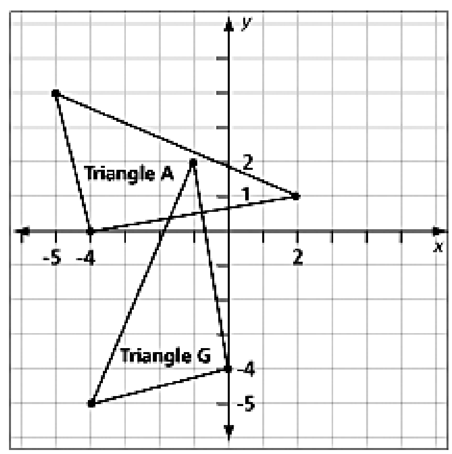
d. Triangle E is obtained by shifting triangle A 1 unit to the left and 2 units up:



e. Triangle F is obtained by reflecting triangle A about the vertical axis and then about the line y = x. Alternatively, it can also be obtained as a -90° rotation of triangle A about the origin (0,0).



f. Triangle G is obtained by reflecting triangle A about the horizontal axis and then about the line y = x. Alternatively, it can also be obtained as a 90° rotation of triangle A about the origin (0,0).



Solutions, cont'd.

Problem H4. For example, (-2,3) becomes (-2,-3). In general, reflecting (x,y) about the horizontal axis yields (x,-y). In other words, the x-coordinate is unchanged while the y-coordinate is the negative of the original y-coordinate.

Problem H5. For example, (-2,3) becomes (2,3). In general, reflecting (x,y) about the vertical axis yields (-x,y). In other words, the y-coordinate is unchanged while the x-coordinate is the negative of the original x-coordinate.

Problem H6. For example (-2,3) becomes (3,-2). In general, if a point (x,y) is reflected about the line y = x, its new coordinates are (y,x). In other words, what used to be the x-coordinate becomes the y-coordinate, and what used to be the y-coordinate becomes the x-coordinate.