Session 10

Classroom Case Studies, Grades 6–8

This is the final session of the *Geometry* course! In this session, we will examine how geometry as a problem-solving process might look when applied to situations in your own classroom. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 6–8 begins below. Watch video programs 11 and 12 in this session. Go to page 215 for grades K–2 and page 229 for grades 3–5.

Key Terms for This Session

Previously Introduced

- congruent
- convex
- Platonic solid
- rectangle
- square
- triangle inequality

New in This Session

- van Hiele levels

Introduction

In the previous sessions, we explored geometry as a problem-solving process. You put yourself in the position of a mathematics learner, both to analyze your individual approach to solving problems and to get some insights into your own conception of geometric reasoning. It may have been difficult to separate your thinking as a mathematics learner from your thinking as a mathematics teacher. Not surprisingly, this is often the case! In this session, however, we will shift the focus to your own classroom and to the approaches your students might take to mathematical tasks involving geometry.

As in other sessions, you will be prompted to view short video segments throughout the session; you may also choose to watch the full-length video for this session. **[See Note 1]**

Learning Objectives

In this session, you will do the following:

- Explore the development of geometric reasoning at your grade level, including the van Hiele model of geometric learning
- Review mathematical tasks and their connection to the mathematical themes in the course
- Examine children's understanding of geometric concepts

Note 1. This session uses classroom case studies to examine how children in grades 6-8 think about and work with geometry. If possible, work on this session with another teacher or group of teachers. A group discussion will allow you to use your own classroom and the classrooms of fellow teachers as case studies to make additional observations.

Part A: Geometry and Reasoning (25 min.)

The study of geometry in high school is often associated with proof—usually an axiomatic development with a focus on two-column, statement-reason type proofs. Newer curricula based on the NCTM standards have reduced this emphasis on two-column proof in geometry in favor of a problem-solving approach. Even so, geometry remains an ideal way to approach reasoning and proof, and it can be started earlier than high school. **[See Note 2]**

When viewing the video segment, keep the following questions in mind:

a. How does the teacher incorporate geometric reasoning into the lesson?

b. Where in the lesson are students learning new geometric content? What is that content?

c. Where in the lesson are students drawing logical conclusions and thinking mathematically? How does the reasoning relate to the geometric content?

d. Thinking back to the big ideas of this course, what are some geometric ideas these students are likely to encounter through their investigation of this situation?

Video Segment (approximate time: 14:07-16:09): You can find this segment on the session video approximately 14 minutes and 7 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, sixth-grade students in Ms. Saenz's class have, through data gathering, conjectured a form of the triangle inequality: Three lengths make a triangle if the sum of any two lengths is greater than the third length. The students, however, are unsure what happens when the sum is equal to the third side. Here, one student tries to explain why he thinks a triangle can't be formed in this case.

Problem A1. Answer the questions you reflected on as you watched the video:

a. How does the teacher incorporate geometric reasoning into the lesson?

b. Where in the lesson are students learning new geometric content? What is that content?

c. Where in the lesson are students drawing logical conclusions and thinking mathematically? How does the reasoning relate to the geometric content?

d. Thinking back to the big ideas of this course, what are some geometric ideas these students are likely to encounter through their investigation of this situation?

Note 2. Before examining specific problems at this grade level, you will watch with an eye toward geometric reasoning a teacher in her classroom who has also taken the course. The purpose in viewing the video is not to reflect on the teacher's methods or teaching style, but to watch closely the way she and the lesson encourage students to tie together their geometric knowledge, intuition, and logical reasoning.

Part A, cont'd.

Problem A2. This lesson is not couched in a "real-world context." Students are thinking about mathematical ideas in the abstract. What are the advantages and disadvantages of this kind of lesson? Are "mathematics only" lessons important in your classroom? What purpose do they, as opposed to contextualized lessons, serve? **[See Note 3]**

Join the Discussion **www.learner.org**

Post your answer to Problem A2 on an email discussion list; then read and respond to answers posted by others. Go to the *Geometry* Web site at www.learner.org/learningmath and find Channel Talk.

Problem A3. Ms. Saenz's lesson was based on a lesson from Session 2, Part B of this course. Discuss the ways Ms. Saenz's lesson was similar to and different from the one in this course. What adaptations did she make and why?

Note 3. This is a particularly good discussion to have with your colleagues. Everyone has different opinions and thoughts about the use of context in the mathematics classroom. Spend some time talking about not just what you think, but why you think it. Cite examples from your own experience instead of focusing on what you have heard others say.

Part B: Developing Geometric Reasoning (40 min.)

Introducing van Hiele Levels

The National Council of Teachers of Mathematics (NCTM, 2000) identifies geometry as a strand in its *Principles and Standards for School Mathematics*.* In grades pre-K through 12, instructional programs should enable all students to do the following:

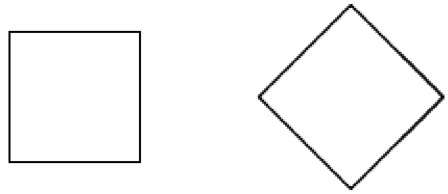
- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Apply transformations and use symmetry to analyze mathematical situations
- Use visualization, spatial reasoning, and geometric modeling to solve problems

In grades 6-8 classrooms, students are expected to do the following:

- Precisely describe, classify, and understand relationships among types of two- and three-dimensional objects using their defining properties
- Understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects
- Create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship
- Describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling
- Examine the congruence, similarity, and line or rotational symmetry of objects using transformations
- Draw geometric objects with specified properties, such as side lengths or angle measures
- Use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involving surface area and volume
- Use geometric models to represent and explain numerical and algebraic relationships
- Recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life

Dutch educators Pierre van Hiele and Dina van Hiele-Geldof developed a theory of five levels of geometric thought. It is just a theory, but a useful one for thinking about activities that are appropriate for your students and prepare them to move to the next level, and for designing activities for students who may be at different levels.

Level 0: Visualization. The objects of thought at level 0 are shapes and what they look like. Students have an overall impression of the visual characteristics of a shape, but are not explicit in their thinking. The appearance of the shape is what's important. Students may think that a rotated square is a "diamond" and not a "square" because it looks different from their visual image of square. *(Early elementary school and, for some, late elementary school)*



* *Principles and Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 2000). Standards on Geometry: Grades 6-8, 41, 232. Reproduced with permission from the publisher.

Part B, cont'd.

Level 1: Analysis. The objects of thought here are "classes" of shapes rather than individual shapes. Students are able to think about, for example, what makes a rectangle a rectangle. What are the defining characteristics? They can separate that from irrelevant information like the size and the orientation. They begin to understand that if a shape belongs to a class like "square," it has all the properties of that class (perpendicular diagonals, congruent sides, right angles, lines of symmetry, etc.). *(Late elementary school and, for some, middle school)*

Level 2: Informal Deduction. The objects of thought here are the properties of shapes. Students begin "if-then" thinking. For example, "If it's a rectangle, then it has all right angles." Students can begin to think about the minimal information necessary to define figures. For example, a quadrilateral with four congruent sides and one right angle must be a square. Observations go beyond the properties into mathematical arguments about the properties. Students can engage in an intuitive level of "proof." *(Middle school and, for some, high school)*

Level 3: Deduction. The objects of thought here are the relationships among properties of geometric objects. Students can explore relationships, produce conjectures, and start to decide if the conjectures are true. The structure of axioms, definitions, theorems, etc., begins to develop. Students are able to work with abstract statements and draw conclusions based more on logic than intuition. *(This is the goal of most 10th-grade geometry courses, but many students are not developmentally ready for it.)*

Level 4: Rigor. The objects of thought are deductive axiomatic systems for geometry. For example, students may compare and contrast different axiomatic systems in geometry that produce our familiar Euclidean plane geometry, finite geometries, the geometry on the surface of a sphere, etc. **[See Note 4]**

For more information on the van Hiele levels and how to work with students within each level, read the article "Geometric Thinking and Geometric Concepts" by John A. Van de Walle from *Elementary and Middle School Mathematics*. This reading is available as downloadable PDF files on the *Geometry* Web site. Go to www.learner.org/learningmath.

Van de Walle, John A. (2001). Geometric Thinking and Geometric Concepts. In *Elementary and Middle School Mathematics: Teaching Developmentally, 4th ed.* (pp. 342-349). Boston: Allyn & Bacon.

Analyzing With van Hiele Levels

In this course, we have primarily worked across levels 2-4. You may feel that the activities we've done are not appropriate for the level of your students, and you're probably right. The goal for this session is for you to think about problems and activities that are at your students' level, and how to help them prepare for the next level of thinking.

In grades 6-8, students should be working comfortably at level 1. Ideally, they will have begun working on drawing logical conclusions and "if-then" thinking characteristic of level 2, but not all students may be comfortable with that kind of task. During middle school, students should be prepared for work at the van Hiele level 3. This means reasoning through more complicated mathematical arguments, leading into some early proofs.

Note 4. If you are working with a group of colleagues, take some time to discuss your own students. Where in the van Hiele levels do you see them functioning comfortably? (There will be a range, of course, because not all students are the same.) Try to cite evidence from you classrooms: What tasks do students find success with? With which tasks do they struggle?

Part B, cont'd.

Video Segment (approximate time: 16:00-23:02): You can find this segment on the session video approximately 16 minutes and 0 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this clip from Ms. Weber's eighth-grade class, the teacher leads the students as a whole class through a proof of the Pythagorean theorem. Students have already reviewed the statement of the theorem, and they have worked through some numerical examples like the one the teacher works through in general. **[See Note 5]**

Problem B1. Where in the video do you see evidence of the following?

- (Level 1 thinking) Students thinking about classes of shapes rather than the individual shapes. Do students seem concerned with orientation or size of the figures?
- (Level 2 thinking) "If-then" reasoning and making geometric arguments
- (Level 3 thinking) Students working more abstractly, drawing conclusions based on logic more than on intuition

Problem B2. Ms. Weber's lesson was based on a lesson from Session 6, Part B of this course. Discuss the ways in which Ms. Weber's lesson was similar to and different from the one in this course. What adaptations did she make and why?

Problem B3. In Session 9, Part A, you worked on the problem of building the five Platonic solids and then arguing from the construction that only five such solids were possible. Recall your own experience in this activity as an adult mathematics learner. During the activity, when did you have to use level 2 thinking? (How did you know when to stop building with triangles and move on to other figures? How did you convince yourself that no other Platonic solids were possible?) What about level 3 thinking?

Problem B4.

a. What do you think were the key pieces of geometry content in this activity? What knowledge did you learn, solidify, or connect with better?

b. What do you think were the key thinking and reasoning skills in this activity? How did the reasoning and geometric content tie together?

Problem B5. Now think about students in grades 6-8 and how this Platonic solids activity might work with them. What must students know and be comfortable with to get the most out of this activity? What are potential stumbling blocks for them?

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Post your answer to Problem B5 on an email discussion list; then read and respond to answers posted by others. Go to the *Geometry* Web site at www.learner.org/learningmath and find Channel Talk.

Problem B6. What might students misunderstand or find confusing in the lesson? How could you alter the lesson or prepare them beforehand to help avoid these misunderstandings?

Note 5. Again, remember that the focus of the video case study is not to examine teaching practice, but to focus on the students and how they are thinking.

Part C: Problems That Illustrate Geometric Reasoning (55 min.)

Geometric Reasoning Problems, Part 1

In this part, you'll look at several problems that are appropriate for students in grades 6-8. As you look at the problems, answer these questions:

a. What is the geometry content in this problem?

b. What skills do students need to work through this problem? What skills will this problem help them develop for later work?

c. What level of geometric thinking is expected of students in the problem? Does it ask students to bridge levels?

d. What other questions might extend students' thinking about the problem?

e. Describe a lesson that you could develop based on the content of this problem. **[See Note 6]**

Problem C1. One way to test whether two figures are congruent is to try fitting one exactly on top of the other. Sometimes, though, it's not easy to cut out or trace figures, so it's helpful to have other tests for congruency.

Each problem below suggests a way to test for the congruence of two figures. Decide whether each test is good enough to be sure the figures are congruent. Assume you can make *exact* measurements. If a test isn't good enough, give a counterexample—that is, an example for which the test wouldn't work.

1. For two line segments, measure their lengths. If the lengths are equal, the line segments are congruent.
2. For two squares, measure the length of one side of each square. If the side lengths are equal, the squares are congruent.
3. For two angles, measure each angle with a protractor. If the angles have equal measures, they are congruent.
4. For two rectangles, find their areas. If the areas are equal, the rectangles are congruent.

Problem C2. Not all of the following statements are true. For the ones that you think are false, make up a counterexample. Then make up two statements of your own, one true and one false.

- If something is a cube, then it is a prism.
- If something is a prism, then it is a cube.
- If something is a square, then it is a rectangle.
- If something is a rhombus, then it is a square.
- All parallelograms have congruent diagonals.
- All quadrilaterals with congruent diagonals are parallelograms.
- If two triangles have the same perimeter, then they are congruent.
- If two rectangles have the same area, then they are congruent.
- All prisms have a plane of symmetry.

Note 6. It's difficult to identify the important content and how students might approach an activity without actually doing the mathematics yourself. These are, for the most part, short problems and activities. Allow yourself time to work through the mathematics, even briefly, before going on to answering the other questions.

Part C, cont'd.

Problem C3. A rectangle has been divided into two congruent parts. What could the parts be?

Geometric Reasoning Problems, Part 2

As you look at the next set of problems, answer these questions:

a. What is the geometry content in this problem?

b. What skills do students need to work through this problem? What skills will this problem help them develop for later work?

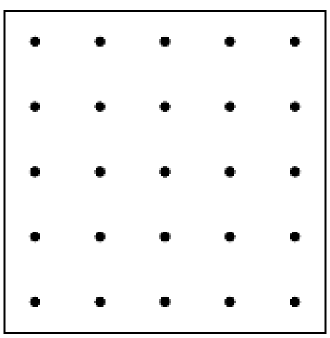
c. What level of geometric thinking is expected of students in the problem? Does it ask students to bridge levels?

d. What other questions might extend students' thinking about the problem?

e. Describe a lesson that you could develop based on the content of this problem.

Problem C4. If a dart has an equal chance of landing at any point on a circular target, is it more likely to land closer to the center or closer to the edge?

Problem C5. On a geoboard, you can make different shapes.



A 5 x 5 Geoboard

1. Make at least five shapes with four boundary pegs and no pegs inside. Find the areas of each of your figures.
2. Make at least five shapes with four boundary pegs and one peg inside. Find the areas of each of your figures.
3. Continue investigating other cases using different numbers of boundary and inside pegs. Can you find a rule for the areas of the figures?
4. Think of a way to explain why your rule works. Hint: What happens when you add a boundary point? How much area is added? What happens when you add an interior point? How much area is added? **[See Note 7]**

Problem C6. Fernando's Frames offers a low-cost, do-it-yourself picture-framing option. To save money, you determine the shape of the frame and select and purchase the side pieces. Side pieces of various lengths are available. Corner fittings are free. Fernando helps you assemble the frame.

One day, Fernando's friend Fred came into the frame shop. He asked for six side pieces, 2, 3, 4, 5, 6, and 7 inches long. He said he could take any three of those side pieces and make a triangular frame.

"Don't be so sure of that!" said Fernando.

What is the probability that any three of these side pieces will form a triangular frame?

Note 7. To make a shape on a geoboard, think about using a rubber band around pegs. You may want to get a real geoboard and work with your colleagues to experiment and gather some data. Remember, you don't want to just think about particular cases such as only examining squares or only examining shapes where the sides are parallel to the sides of the geoboard.

Homework

Solutions are not provided for these homework problems, since answers will vary depending on individual experiences.

Problem H1. Interview a teacher in the grade level above you. Pick one of the problems in this session, and ask him or her the following questions:

a. How does the content of this problem prepare students for geometric thinking in your grade?

b. Why do you think this content is important?

c. How could this problem be extended for students in your grade?

Problem H2. Look at a lesson or activity in your own mathematics program for your grade level that you think has potential for developing students' geometric reasoning. If you were to use this lesson or activity now, after taking this course, how might you modify or extend it to bring out more of the important ideas about geometry?

Solutions

Part A: Geometry and Reasoning

Problem A1. Answers will vary. Some ideas: Students come up with the triangle inequality through data gathering and checking, but the teacher persists in asking them to explain why it should hold. That stretches students into not just the geometry content, but also the logical deductions. This course covered the triangle inequality, and several of the same ideas came up. One particularly interesting piece is at the beginning of the clip, when students use a geometric model to check if the sum of two lengths is greater than the third length. Many teachers rely on algebra and arithmetic more, and it's nice to remember that many students do have a sense of geometric reasoning as well.

Problem A2. People have very different, and often very strong, opinions about the use of context in mathematics classrooms. It is important to present students with a variety of lessons. Students can be engaged by problems that are not context-based, as well as by those with real-world connections.

Problem A3. There were many adaptations. Here are some: The materials were different; Ms. Saenz used what was available at her school. The activity was more directed; students were asked to look particularly at sums of sides rather than to find whatever relationship they could.

Part B: Developing Geometric Reasoning

Problem B1. Answers will vary. Some possible responses:

- (Level 1 thinking) The students easily calculate the areas of the squares and triangles in different positions and recognize the triangles as congruent even though they are positioned differently.
- (Level 2 thinking) By summing areas to find the total and equating the areas found in two different ways, students are showing logical thinking about geometric objects.
- (Level 3 thinking) This is harder to see. The teacher is clearly trying to move them through a multi-step argument, but students may not all be aware that they are using previous results (area formulas, algebraic facts, solving equations) to prove something new.

Problem B2. There were many adaptations. Here are some: The proof was adapted to be one that was easier for students to follow. (It was based on the Garfield proof in this course—see Session 6, Part B—but was adapted to remove the need for computing with fractions.) Students worked through numerical and numeric/variable mixed problems before working with variables only. The teacher works with students as a whole class (on the proof itself) rather than asking them to work through individually or with pairs.

Problem B3. Answers will vary. The thinking often goes like this: "If it's going to make a solid shape, then there must be at least three polygons meeting at a vertex. If it's going to make a solid shape, then there must be less than a total of 360° around a vertex. Regular hexagons and polygons with more than six sides all have 120° or more at each vertex, so these shapes cannot be used." And so on. Putting all of this together to convince yourself that only five such solids are possible constitutes level 3 thinking.

Problem B4. Answers will vary. Some possible answers:

a. Key pieces of geometry are definitions and properties of regular two- and three-dimensional figures, building polyhedra, angle relationships, and so on. We also explored Euler's formula in this course (see Session 9, Problem A8).

Solutions, cont'd.

Problem B4, cont'd.

b. "If-then" thinking, reasoning through every possible case, and generalizing were all important parts of the activity. It was important to both know the geometry (what are the angle measures for polygons with different numbers of sides?) and to use those facts in making deductions.

Problem B5. Answers will vary. Students will probably gain understanding of three-dimensional figures and how they differ from polygons. They will likely gain valuable understanding and visualization skills from building and manipulating the solids and from attempting to count faces, edges, and vertices. They may not have the prerequisite knowledge of angle measures in polygons as a solid foundation. Some students may also struggle with the generalizations. If six triangles don't work, how do we know seven triangles won't work? Why can we eliminate polygons with seven, eight, and more sides without even trying to build them?

Problem B6. Answers will vary. Some ideas: Lots of experience with building generalizations in cases that are easier to check, and lots of experience with polygons will help.

Part C: Problems That Illustrate Geometric Reasoning

Problem C1.

a. The content in this problem covers congruence, properties of figures, and also ideas such as counterexample and testing conjectures. Students will be able to reason through different properties and examine which one will be sufficient to assure congruence. Unlike parts (a)-(c), in part (d) they will discover that having the same area does not ensure congruence of the shapes. In other words, areas can be the same for non-congruent shapes, even if they belong to the same class of shapes.

b. To work through this problem, students should already understand the idea of congruence well and should be familiar with all of the shapes and properties named. In this course, we looked at some triangle congruence and similarity tests, and this is a similar activity for other kinds of figures. This prepares students both for moving into the triangle congruence tests and for working more with proof and counterexample. It helps them test conjectures because they have to come up with cases to test themselves.

c. This requires a combination of level 1 (thinking about properties of figures, reasoning about same class of shapes) and level 2 (thinking about minimal information necessary to define and test congruence; also, "if-then" thinking—e.g., if two shapes have the same area, they may or may not be the same shape, and consequently may or may not be congruent).

d. To extend students' thinking, you could ask them to come up with their own congruence tests for some shapes (for example, circles or cubes) and to think creatively about different tests that would work. For example, if you had two boxes, and you had sand (to fill them), paper (to wrap around them), and string, what are some ways you could test the two boxes for congruence? What would be enough to convince you one way or the other?

e. One possible lesson: Begin with a short activity that reviews congruence as "shapes fitting on top of each other." For example, give each student a sheet with three different pictures, and ask them to find their "buddy"—the student with three pictures on their page congruent to the ones on yours. Each student will need a sheet with three pictures. You can vary from eight or so pictures, changing the orientations and size of the pictures so that only two students (or three students, or however many you choose) have matching sheets. You can use this to form groups for the day, or just as a warm-up. Review the idea of congruence as "same size and shape," but explain that sometimes it's difficult or impossible to try to fit the shapes on top of each other to check for congruence. Pass out sheets with the congruence tests. Ask students to create examples for each congruence test. (For example, use grid paper to draw several different rectangles with the same area.) Then decide if all the examples they drew are indeed congruent. End the activity by discussing each test, asking students to provide their test cases, and talking about how to come up with good test cases. (Slanted orientations, long and skinny rectangles, and other "extreme cases" should be the focus.)

Solutions, cont'd.

Problem C2.

a. This task is focused on different types of "if-then" thinking. Some of the statements are based on definitions and are fairly straightforward. Others are about families of figures, and still others require more logical deduction; e.g., "If two rectangles have the same area, then they are congruent."

The geometric content is different depending on the statement students investigate. For example, the statement "If two triangles have the same perimeter, then they are congruent" is about congruence of triangles. The statement is false. The counterexample is that it is possible to have two triangles of the same perimeter that are not congruent—for example, triangles whose sides are 5, 5, 2 and 4, 4, 4. These are isosceles and equilateral triangles, respectively, and they are not congruent. A true statement may be the following: "If two triangles have all three sides the same length, then they are congruent." A false statement may be as follows: "If two triangles have the same perimeters, then their areas are also the same." Notice the content area is no longer the congruence of triangles, but rather the concept of area and perimeter of triangles.

b. To work through this task, students should be familiar with all of the terms used so that they do not struggle when coming up with examples to test. They should also have experience in other activities of coming up with examples and non-examples to fit definitions. This activity prepares students for work on mathematical proof. Here, students must decide on the veracity of a statement (a key first step in proof), and come up with counterexamples to false statements (an essential proof technique).

c. This problem is similar to the definition activities we did in Session 3, Part C. This is a level 2 task, with a focus on "if-then" thinking and simple deductions. Students come up with their own test cases and thus make generalizations, which they compare to the given statements and make further deductions.

d. To extend students' thinking, you can vary the types of "if-then" statements used. You can also use activities like this to introduce and explore the idea of a converse (switching the "if" and "then" parts of a statement). Sometimes the converse of a statement is true, and sometimes not. Examples:

- Statement: "If a polygon has three sides, then it is a triangle."

  True converse: "If a polygon is a triangle, then it has three sides."

- Statement: "If a quadrilateral is a square, then all of its sides are the same length."

  False converse: "If a quadrilateral has all sides the same length, then it is a square."

You can use non-mathematical situations to explore the idea of converse further, and perhaps make it even clearer. Examples:

- Statement: "If you live in New York City, then you live in New York state."

  False converse: "If you live in New York state, then you live in New York City."

- Statement: "If an animal has feathers, then it is a bird."

  True converse: "If an animal is a bird, then it has feathers."

e. One possible lesson: Choose one of the false statements; for example, "If two triangles have the same perimeter, then they are congruent." Tell students they have five minutes to decide if it is true or false and to find a way to convince you and their classmates that they are right. Encourage them to draw or write down their ideas. When it seems most students have decided, ask one or two students to answer true or false and explain how they decided. Introduce or review the word "counterexample" when a student shows an example of two triangles with the same perimeters that are not congruent. Then ask students to, on their own, come up with a statement like that one, but that they are sure is true. Ask a few students to share their statements, and how they know they are right. (For example, students might say, "If two triangles have the exact same side lengths, then they are congruent," because that is a triangle congruence test that they know. Or they might create the converse of the given statement, "If two triangles are congruent, then they have the same perimeter," explaining that if they're congruent, each of the three sides have the same length, so the total is the same.) Then pass out several sheets of paper. Each sheet should have just one "if-then" statement on the top, with room for students to draw examples and to write their conclusion. They are to decide if the statement is true or false and explain why.

Solutions, cont'd.

Problem C3.

a. The content in this problem covers properties of figures and congruence. It also requires visualization and thinking through possible cases.

b. To engage in the task, students should understand congruence and what a rectangle is.

c. This is a level 1 problem, with the possibility for extensions to level 2 thinking. The students begin by thinking about and analyzing all types of rectangles, not just one. They then move into "if-then" type of thinking. For example, if the two new shapes are to be congruent, then they must have all side lengths the same. It would be interesting here to explore why cutting a rectangle into two shapes with the same area would not be enough to ensure they are congruent shapes. Writing up a careful solution of all possible cases (including starting with special rectangles) and explaining why it's a complete list would be a level 3 task.

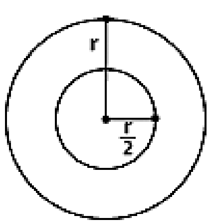
d. The problem could be further extended by asking questions like, "I cut my rectangle into two congruent squares; what can you say about my original rectangle?" You could also think about equal area rather than congruence, and cutting into more than just two pieces.

e. One possible lesson: You could use this as an introduction to a longer activity. Hold up a regular sheet of paper and say: "I want to divide this into two congruent pieces. What can I do?" Take some suggestions from students, try each one, and test (by fitting the figures on top of each other) if it results in congruent pieces. Tell students that you want to divide the sheet of paper into two congruent squares and get their reactions. When they decide it is impossible, ask them on their own to come up with a rectangle that can be divided into two congruent squares. Then pass out several sheets of paper and scissors to each student. Their goal is to find every possible way to divide the rectangle into two, three, and then four congruent pieces. You could also have different students or groups work with different starting shapes to see if some figures give more options for results than others.

Problem C4.

a. The content here is areas of circles (and parts of circles) and probability. The nice thing is that it's a vague statement that students need to clarify. What does "closer" mean? Answering this question requires calculating which area is larger and thus would have a larger chance of being hit.

   In this case, the total area can be divided into two areas—closer to the center and closer to the edge. The area closer to the edge will be larger and thus more likely to be hit. Here's how it can be done. Draw a circle. Draw another one inside of it, with the same center and a radius half as big. If the dart lands inside this smaller circle, it's closer to the center. If it lands outside the smaller circle, it lands closer to the edge. The area of the smaller circle is $(\pi r^2)/4$. So the area of the other shape (it's called an annulus) is πr^2 minus the little circle, or $(3/4)\pi r^2$. So it's three times as big as the little circle, and three times as likely to be hit.



b. This is very similar to the multi-step geometric problem solving that will be asked of students in high school and beyond. It helps them practice skills of reasoning through difficult problem statements, drawing pictures of situations, and applying knowledge (areas of circles, for example) in unfamiliar contexts.

c. This is a level 2 task; it asks students to use their knowledge of calculating areas to draw conclusions about probability. This problem also asks students to apply knowledge of probability to solve a geometric problem.

Solutions, cont'd.

Problem C4, cont'd.

d. Ways to extend this task include using different-shaped dartboards, creating a dartboard where points are assigned based on difficulty of hitting the region, and creating fair probability games. (For example, you want a dartboard divided into two regions—a circle and an annulus like above—but you want an equal probability of hitting each piece. Where should you divide the circle? Does it depend on the size of your starting circle?)

e. One possible lesson: You can use a magnetic dartboard or similar toy to start a game. Divide the class into two teams and have each team send a representative who will take turns throwing the darts. The rule is, if the dart lands closer to the center, team A gets a point. If it lands closer to the edge, team B gets a point. If it doesn't hit the board, it's a do-over. Let each team take five to 10 tries. The class will have to decide how to measure distance to the center and distance to the edge, though you can lead them to the idea of measuring along a radius that passes through that point. Total up the points, and ask if you think it was a fair game. Get lots of responses, and take some notes of students' ideas on the board. Ask students what calculations they could make to decide if it really is a fair game, and guide the discussion to the idea of comparing areas. Review the area formula for a circle, and then give students some time to make whatever calculations they think are appropriate. End with a discussion in which students share their ideas and in which someone (a student or you) shows a picture like the one above and explains how to calculate the relevant areas. You can end there, or with the question of how to reposition the dividing line to make the game fair.

Problem C5.

a. This problem contains information about areas, asking students to develop a formula that depends not on the kind of shape (i.e., how many sides it has), but instead on its configuration on a geoboard. The relationship that they discover is between the interior points, exterior points and the area of a shape. The formula, known as Pick's theorem, follows:

   Let P be a lattice polygon. Assume there are I(P) lattice points in the interior of P and B(P) lattice points on its boundary. Let A(P) denote the area of A. Then A(P) = I(P) + B(P)/2 - 1.

   So, for example, using this formula, the area of a square that includes four boundary pegs and no pegs inside will be A = 0 + (4 / 2) - 1 = 1. This is just what you would expect.

b. To engage in the task, students should have a solid understanding of forming different shapes on a geoboard and calculating their areas. They should be able to find the areas of long skinny triangles and of tilted squares, as well as the areas of shapes in standard positions. Students should also have experience gathering data, organizing it in a table, and generalizing from patterns.

c. The initial investigation is a level 2 task, asking students to come up with cases to check and develop rules like, "If it has four boundary points and one interior point, then the area is 2 no matter what the shape is." Using the induction-like argument suggested in part 4 of the problem moves this to a solid level 3 task, where students must think to divide a bigger shape into smaller components, about which they already know the areas.

d. This problem moves students toward thinking about more complicated numerical patterns and problem situations, since there are two variables (boundary pegs and interior pegs) that need to be dealt with separately. It introduces students to the very important idea of controlling variables, allowing only one thing to change at a time so you can see how different changes affect the situation. (Note that the problem itself suggests ways for extending students' thinking, especially in part 4, which moves them toward the idea of proof by induction.)

Solutions, cont'd.

Problem C5, cont'd.

e. One possible lesson: On a demonstration geoboard or on the blackboard, draw several shapes and explain how to count boundary pegs and interior pegs. Then ask students to tackle part 1 of the problem (making shapes with four boundary pegs and no interior pegs). Give them several minutes for the task; then bring the class together to share results. They should have found that all of the shapes—triangles, squares, and parallelograms—had an area of exactly one. Make sure to show several different examples of each type of shape, including strangely oriented ones, to get students thinking about those types of figures. Ask students to conjecture whether shapes with four boundary pegs and one peg inside will have more or less area than the ones with just four boundary pegs. Also, will they all have the same area? Put some conjectures on the board, and then again ask the students to investigate it. Bring the class back together and discuss their results. Start a table on the board like this:

B (Boundary Pegs)	I (Interior Pegs)	A (Area)
4	0	1
4	1	2

Tell students that their job is to continue the table, create several examples of each case, and come up with a formula for the area based on boundary and interior pegs. Let students work for a long time on this task. Depending on how much structure they need organizing their work, you may want to provide them with a table in which numbers of boundary and interior pegs are filled in, so that it is set for them to only change one variable at a time. Alternately, you may just want to make that important strategy clear at the outset and leave it to students to organize the work themselves. Wrap up the activity by extending the table on the board (with students' help) to several other cases, and writing a formula. Depending on the class, you may want to work with them on parts of the explanation suggested in part 4 of the activity.

Problem C6.

a. The underlying content here is the triangle inequality, which states that if the sum of any two lengths is greater than the third length, then the three lengths will make a triangle. Again, there is some elementary probability involved. The problem requires students to think systematically about sets of three lengths and to find a way to know they have considered all of the possibilities. The total number of possible cases is 20. Seven combinations will not make a triangle, so the probability there is 7/20. Therefore, 13 combinations will make a triangle (for a probability of 13/20, or 65%).

b. To engage in the task, students should already know the triangle inequality (though they may have forgotten its exact statement), and they should know how to calculate probabilities.

c. The thinking skills necessary to look at this problem move it into a level 2 task. Students explore the relationships between different side lengths and produce conjectures about probability. Again, they utilize the "if-then" type of thinking; e.g., if the sum of two sides is greater than a third one, it will make a triangle. This task prepares students for later work in which they must bring different areas of mathematical knowledge together; for example, using their algebra skills in a geometry problem, and here combining geometry and probability. It is also good for them to practice longer word problems, where they must decipher what the problem is and what the relevant information is.

d. Questions that could extend students' thinking include the following: Come up with a set of six lengths so that any three do in fact make a triangle. Come up with a set of six different lengths so that any three do in fact make a triangle. Come up with a set of six lengths so that no set of three of them will make a triangle.

Solutions, cont'd.

Problem C6, cont'd.

e. One possible lesson: Pass out the problem; give students a moment to read it over, and encourage them to underline anything they think is particularly important to the problem. Then ask one or more students to summarize the problem: What are we trying to answer? Ask students to help you formulate a question to write on the board and to decide on the information that will help you answer it. Pass out materials such as linkage strips or snap cubes, and ask students to work on the problem. After several minutes, bring the class together and ask them what they've found. Did anyone find a set of three lengths that didn't work? What went wrong? Now is the time to review the triangle inequality and to write down the statement on the board. Review what students will need to know to answer the problem—how many combinations of three lengths are possible and how many of them do in fact make a triangle. Then ask students to resume the task, keeping these two goals in mind. Complete the activity by asking students to share their probabilities. It's likely that students missed some of the total combinations as well as some of the working combinations, so you may want to model for them how to make an organized list of possibilities, ensuring you don't miss any, and then how to check each one for satisfying the triangle inequality.