Session 5

Dissections and Proof

Key Terms for This Session

Previously Introduced

- midline
- rectangle
- trapezoid
- parallelogram
- side-angle-side (SAS) congruence
- vertex
- quadrilateral
- square

New in This Session

- midline theorem
- tangram
- reflection
- translation
- rotation

Introduction

In this session, you will begin to use the properties of figures to solve geometric problems and to justify their solutions. You will also use visualization and geometric reasoning to solve cutting tasks.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will learn to do the following:

- Dissect (cut) geometric figures into pieces you can rearrange to form different geometric figures

- Use geometric language to describe your reasoning, justifications, and solutions to problems

- Prove the midline theorem

Note 1.

Materials Needed:

- tangram set, or a large square of stiff construction paper so that you can make your own tangram set (see Homework for instructions)
- copies of the figures to be cut out for Part B (they can be photocopied to make additional copies)
- blank white paper (at least 12 pages)   • scissors   • ruler
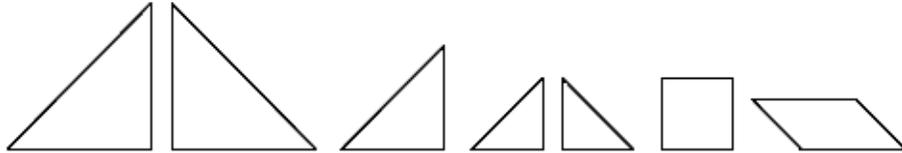
Tangram Set
You can purchase tangram sets from the following source:

ETA/Cuisenaire
500 Greenview Court
Vernon Hills, IL 60061
800-445-5985/847-816-5050
800-875-9643/847-816-5066 (fax)
http://www.etacuisenaire.com

Part A: Tangrams (15 minutes)

Exploring Tangrams

A tangram is a seven-piece puzzle made from a square. A typical tangram set contains two large isosceles right triangles, one medium isosceles right triangle, two small isosceles right triangles, a square, and a parallelogram.

Use a set of tangrams, or cut out the set provided on page 116, to work on the following problems.

Problem A1. Given that the tangram puzzle is made from a square, can you recreate the square using all seven pieces? **[See Tip A1, page 119]**

Problem A2. Use all seven tangram pieces to make a rectangle that is not a square.

Moving Tangrams

One set of tangram pieces can make more than one shape. Geometric language helps you give clear descriptions of how to move pieces to change one shape into another. You will use three basic moves:

Slide or Translation:

Here, a parallelogram made of two small triangles becomes a square as you slide one of the small triangles parallel to the base of the parallelogram. This type of movement is known as a translation. Try this with your tangram pieces.
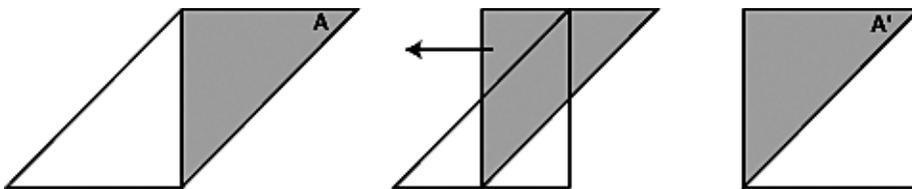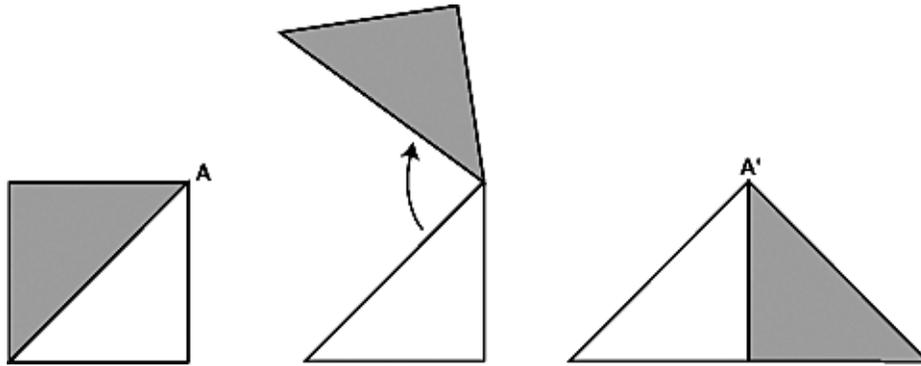
Flip or Reflection:

Here, the parallelogram becomes a triangle when you flip one of its halves (small triangles) at its horizontal midline. This type of movement is called a reflection. Try it.
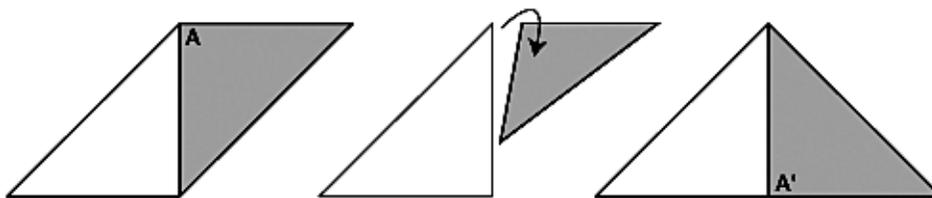
Part A, cont'd.

Rotation:

Here, a square becomes a triangle when you rotate one of its two halves (small triangles) 270° around vertex A. This type of movement is called a rotation. Try it.

Problem A3. For each of the pairs of figures below, do the following:

- Build the shape on the left with your tangram set.

- Turn it into the shape on the right by reflecting, rotating, or translating one or two of the pieces. (This may take more than one step.)

- Write a description, telling which piece or pieces you moved and how you moved them.
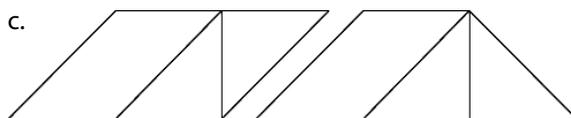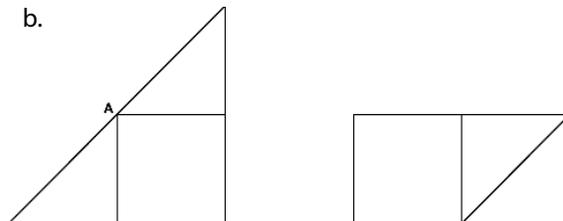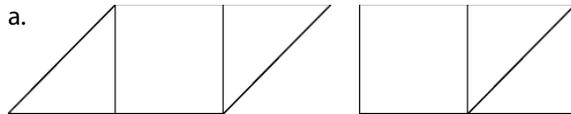
a.

b.

c.

Part B: Cutting Up (50 minutes)

You can solve each of the following problems in more than one way. Try to find several solutions; then pick the one that you like best. Jot down notes about each solution so that you can reconstruct your work later. Use the figures on page 117 in Problems B1–B6.

For each problem, start with the figure described. Find a way to cut that figure into pieces you can rearrange to form the second figure described. You have to use all the pieces from the original figure and put them together like a puzzle—no gaps or overlaps are allowed. Pay attention to how you can justify your cutting process. Use the properties of your beginning figure and the cuts you made to explain why you ended up with the second figure. **[See Note 2]**
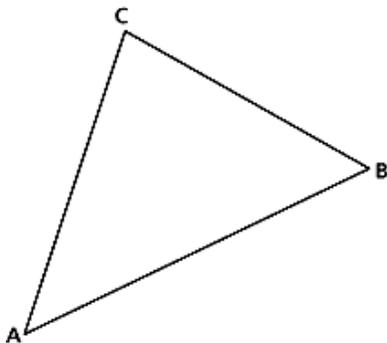
Problem B1. Start with a parallelogram. Find a way to cut your parallelogram into pieces you can rearrange to form a rectangle.
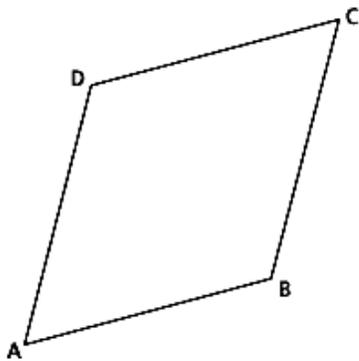
Problem B2. Start with a right triangle. Dissect the triangle so that you can rearrange the pieces to form a rectangle.
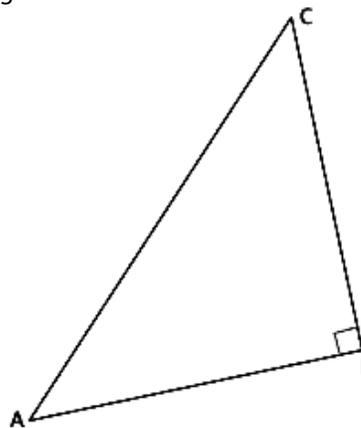
Problem B3. Start with a scalene, non-right triangle. Cut it into pieces that will form a parallelogram.

Note 2. If you are working in a group, pose the first problem and give everyone a couple of minutes to work on it. Watch for someone with an appropriate solution. Ask that person to share the solution and explain why it works. That's the goal for these problems: to come up with methods that will always work and that don't rely on measurement. Also, make sure that everyone is clear about the "rules of the game," particularly that you have to use all the pieces and you have to be able to describe your method using the language of flip, rotate, and translate. At the end, leave at least 10 minutes to share methods that can be explained, as opposed to those that just look right.

Part B, cont'd.

Video Segments (approximate times: 5:45-7:23 and 8:52-9:52): You can find the first segment on the session video approximately 5 minutes and 45 seconds after the Annenberg/CPB logo. The second segment begins approximately 8 minutes and 52 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, groups of participants work to transform several shapes into new figures. As they progress towards more difficult problems, they are able to make increasingly complex justifications for why their cuts work.
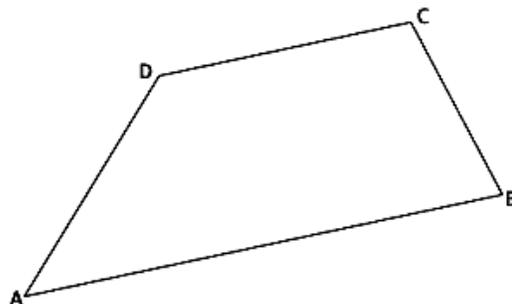
Were you able to make similar kinds of justifications?

Problem B4. Start with a scalene, non-right triangle. Cut it into pieces that will form a rectangle. **[See Note 3]**

[See Tip B4, page 119]
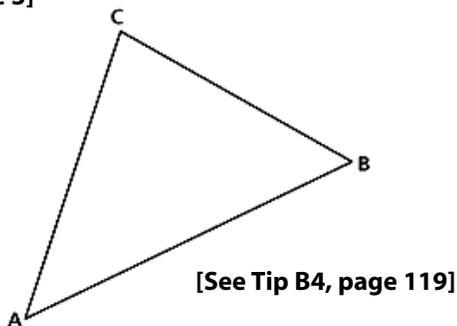
Take It Further

Problem B5. Start with a trapezoid. Dissect the trapezoid into pieces that will form a rectangle.

Take It Further

Problem B6. Start with a trapezoid. Dissect the trapezoid into pieces that will form a triangle. As a challenge, see if you can do this with a single cut.

Take It Further

Problem B7. In several of the problems above, you had a rectangle as a final result. For each of the following shapes, start with a rectangle and dissect it into pieces you can rearrange to form that shape:

    a. An isosceles triangle

    b. A right triangle

    c. A scalene triangle

    d. A non-rectangular parallelogram

    e. A trapezoid

Note 3. After you have solved Problems B1-B4, reflect on why your cuts work. It's likely that you used something like a "midline cut" to turn a triangle into a parallelogram and then into a rectangle. Consider that we want a single cut to do two different things: connect consecutive midpoints (so that the segments you attach will match up), and form a segment that is parallel to the base (so that the final figure will have parallel sides). This leads into Part C, where we prove that these two segments really are one and the same.

Problems B1-B5 are adapted from *Connected Geometry,* developed by Educational Development Center, Inc., p. 164.
© 2000 Glencoe/McGraw-Hill. Used with permission. www.glencoe.com/sec/math
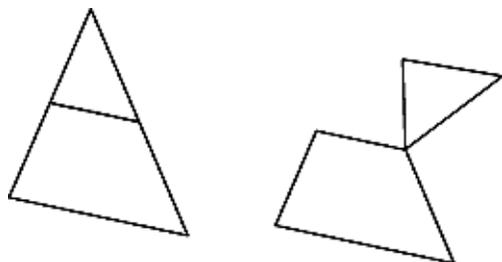
Part C: The Midline Theorem (55 minutes)
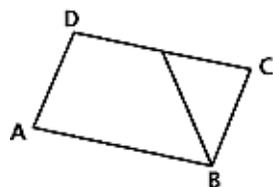
The Midline Cut

In this part, we will begin to use segment notation. Optional: See "About Segment Notation" (page 118).

Let's look again at how we solved Problem B3, in which we dissected a triangle to form a parallelogram. **[See Note 4]**

Find the midpoints of two sides of a triangle. Cut along the segment connecting those two midpoints.

Rotate the top triangle 180° about one of the midpoints.

The two segments match because the cut was at the midpoint. The following are the conjectures that we will prove in the midline theorem:

- Quadrilateral ABCD is a parallelogram because the opposite sides are the same length. AD and BC are the same length because they were made by cutting at a midpoint.

- AB and CD are the same length because a midline cut makes a segment half as long as the base.

Note 4. If you are working in a group, check to see if anyone used a cutting method similar to the one presented here. If so, consider using their method to launch the proof rather than the one shown here.
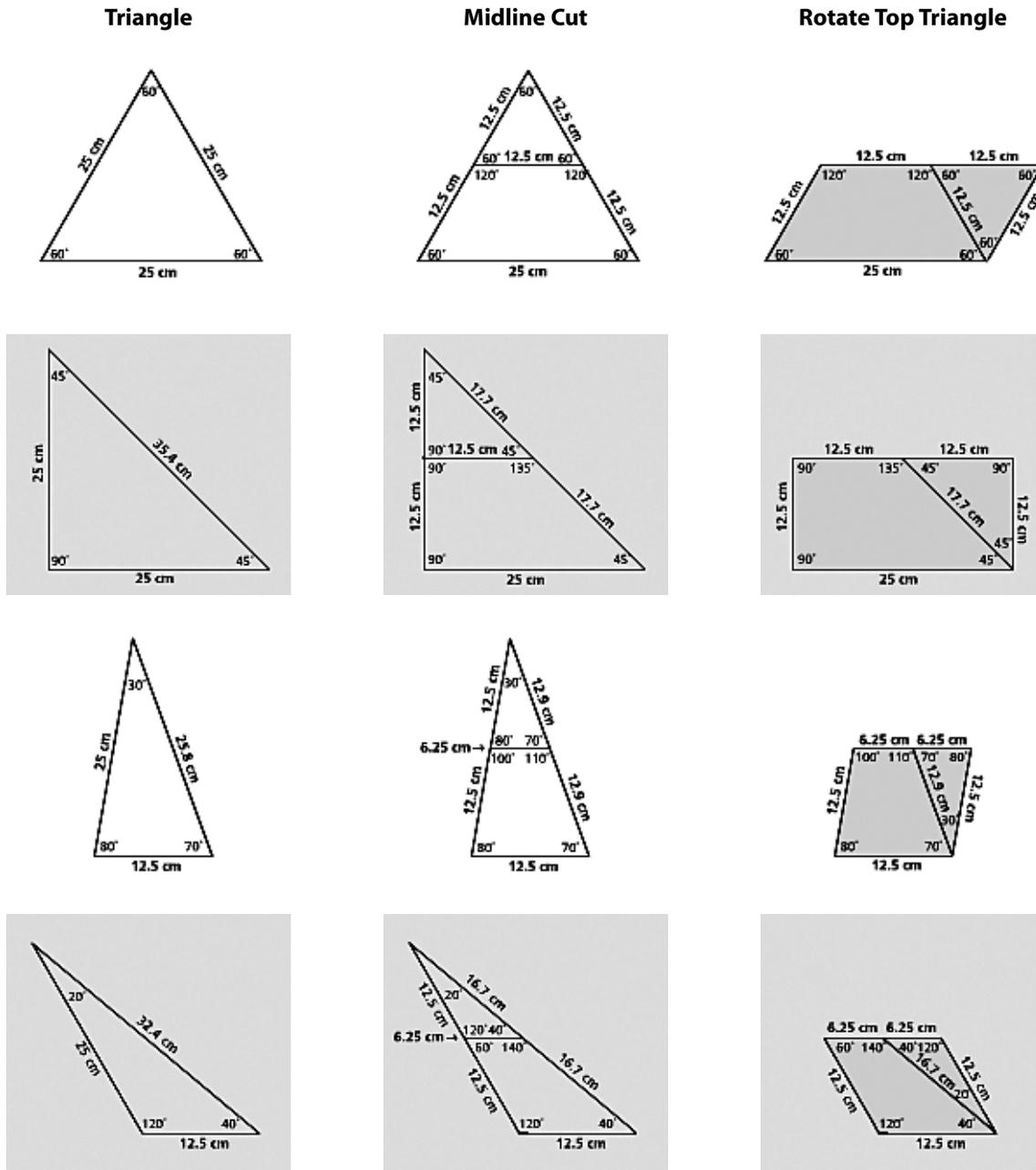
Part C, cont'd.

Here are examples of midline cuts for four triangles (regular, right, isosceles, obtuse):

| Triangle	Midline Cut	Rotate Top Triangle

Part C, cont'd.

Some Geometry Facts

The midline theorem claims that cutting along the midline of a triangle creates a segment that is parallel to the base and half as long. Does that seem reasonable?

To prove the midline cut works, you need to use some geometry facts that you may already have encountered. If not, take some time to consider why these statements are true. **[See Note 5]**

Fact 1: Vertical angles (the angles opposite each other when two lines intersect) are congruent (they have the same measure).
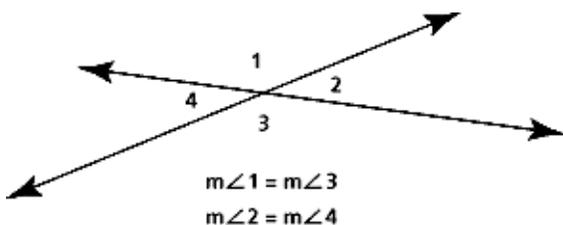
$$m\angle 1 = m\angle 3$$
$$m\angle 2 = m\angle 4$$

Why? We can show why, for example, $m\angle 1 = m\angle 3$:

$m\angle 1 + m\angle 2 = 180°$ and $m\angle 2 + m\angle 3 = 180°$, since in both cases the two angles together create a "straight angle." So $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3 = 180°$. Subtracting $m\angle 2$ from each part of the equation, we see that $m\angle 1 = m\angle 3 = 180° - m\angle 2$.

Fact 2: If two triangles have two sides that are the same length, and the angle between those two sides has the same measure, then the two triangles are congruent. The two triangles must have the same size and shape, so all three sides have the same length, and all three angles have the same measure. This is known as SAS (side-angle-side) congruence.
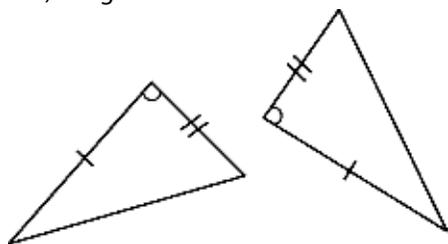
The single tick indicates the two sides that are the same length. The double tick indicates the two sides that are the same length. The angle markings indicate that those two angles have the same measure.

Why? The easiest way to be convinced of the fact that the two triangles are congruent is to draw some triangles. Draw a segment two inches long and a segment three inches long, with a 60° angle between them. Is there more than one way to complete the triangle? Come up with other cases to try.

Note 5. Spend a few minutes reflecting on or discussing each fact. If you are working in a group, it may help to have posters of these that everyone can refer to as they work on the proofs.

Part C, cont'd.

Fact 3: There are many equivalent definitions of "parallelogram":

   · A quadrilateral with both pairs of opposite sides parallel

   · A quadrilateral with both pairs of opposite angles congruent

   · A quadrilateral with both pairs of opposite sides congruent

   · A quadrilateral with one pair of opposite sides both congruent and parallel

The last definition is the one that will come in handy here. Go through the steps of understanding a definition in Session 3 if you're not sure why it works.

Proving the Midline Theorem

With these facts in mind, let's prove that the midline cut works.

Cutting along the midline of a triangle creates a segment that is parallel to the base and half as long.

Start with this picture. It was created so that D and E are midpoints of AC and BC respectively. D, E, and F are collinear (on the same line), and DE and EF are the same length. **[See Note 6]**
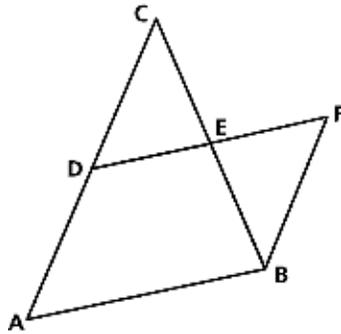

Problem C1. Looking at the facts presented on pages 110-111, explain why triangles DEC and FEB are congruent.

Problem C2. Explain why AD and BF have the same length.

Note 6. It's sometimes hard for people to separate this figure from the cutting activities they've just been engaged with. Make sure you're clear about how the figure was constructed; it might help to create it yourself. First, start with the triangle. Find midpoints D and E and connect them. Then extend that segment to twice its length. Recall what we know: AD = DC (D is a midpoint); BE = EC (E is a midpoint); DE = EF (we made the figure that way); angle DEF is 180° (we made it a straight line). That's all we know.

Part C, cont'd.

Problem C3. Explain why AD and BF are parallel. **[See Tip C3, page 119]**
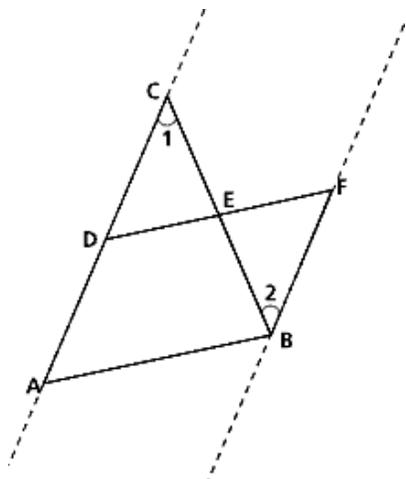
Problem C4. Explain why ABFD is a parallelogram.

Problem C5. Explain why DE is parallel to AB and half as long.

By proving the midline theorem—since we proved it, we can call it a theorem now—we illustrate some important mathematical ideas.

- First, you know that the midline cut will always work, on any triangle. There is nothing special about triangle ABC in the picture. So now, no matter how long and skinny, or how funny-looking a triangle you get, you can be sure the midline theorem will hold. You could never test it out on every triangle because there is an infinite number of cases to try, so a proof is the only way to be sure.

- Second, now that you're sure the midline theorem holds, you can use it when you look at other problems.

Video Segment (approximate time: 23:29-25:21): You can find this segment on the session video approximately 23 minutes and 29 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Being able to make these kinds of proofs can be a useful skill in the professional world. In this segment, Mr. Ialeggio explains how, in order to ensure accurate dimensions, he proves that the window frames he makes for antique homes are indeed rectangles.

Take It Further

Problem C6. In the picture at right, AB equals BC, and CD equals AD; and E, F, G, and H are the corresponding midpoints. Explain why EF is parallel to HG and why they are the same length. What kind of quadrilateral is EFGH?

[See Tip C6, page 119]
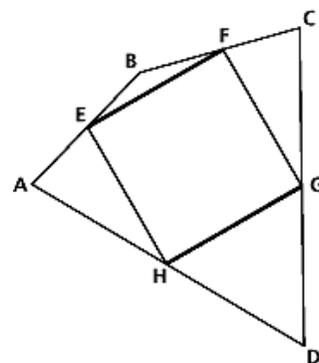
Take It Further

Problem C7. Draw another quadrilateral and connect the midpoints of the four sides in order. What kind of quadrilateral is formed and why? **[See Note 7]**


Note 7. You may have already made a conjecture about this situation in the Take It Further section of Session 4. Now you have the tools to actually prove it.

Problem C1-C6 are adapted from *Connected Geometry,* developed by Educational Development Center, Inc., pp. 194-195. © 2000 Glencoe/McGraw-Hill. Used with permission. www.glencoe.com/sec/math

Homework

Problem H1. You can make your own tangram set from construction paper. Start with a large square of construction paper and follow the directions below:

Step 1: Fold the square in half along the diagonal; unfold and cut along the crease. What observations can you make about the two pieces you have? How could you prove that your observations are correct?
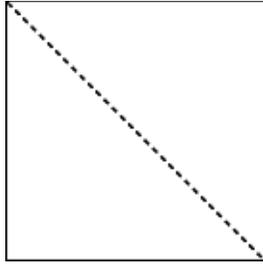
Step 2: Take one of the triangles you have, and find the midpoint of the longest side. Connect this to the opposite vertex and cut along this segment.
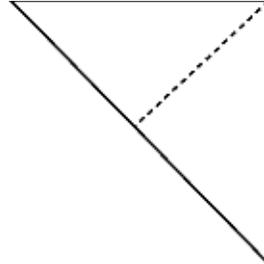
Step 3: Take the remaining half and lightly crease to find the midpoint of the longest side. Fold so that the vertex of the right angle touches that midpoint, and cut along the crease. Continue to make observations.
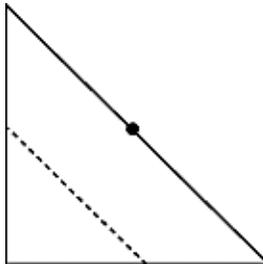
Step 4: Take the trapezoid and find the midpoints of each of the parallel sides. Fold to connect these midpoints, and cut along the fold.
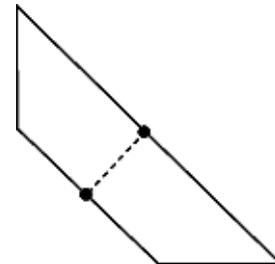
Step 5: Fold the acute base angle of one of the trapezoids to the adjacent right base angle and cut on the crease. What shapes are formed? How do these pieces relate to the other pieces?
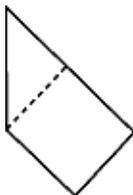
Step 6: Fold the right base of the other trapezoid to the opposite obtuse angle. Cut on the crease. You should now have seven tangram pieces. Are there any other observations you can make?
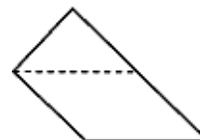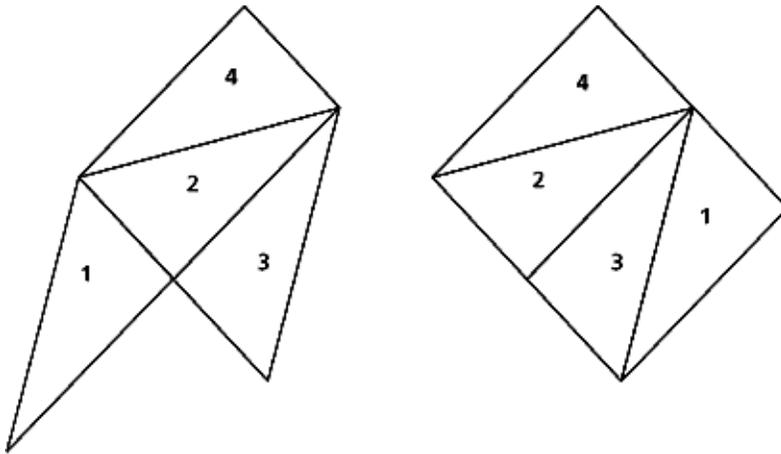
Homework, cont'd.

Problem H2. When people work on the dissection problems, they often create a figure that looks like a rectangle, but they can't explain why the process works. Or sometimes they perform a cutting process that they think should work, but the result doesn't look quite right. Reasoning about the geometry of the process allows you to be sure. A cutting process is outlined below. Your job is to analyze if the cuts really work. Does this algorithm turn any parallelogram into a rectangle? If so, provide the justifications. If not, explain what goes wrong.

First, cut out the parallelogram. Then fold along both diagonals. Cut along the folds, creating four triangles as shown.
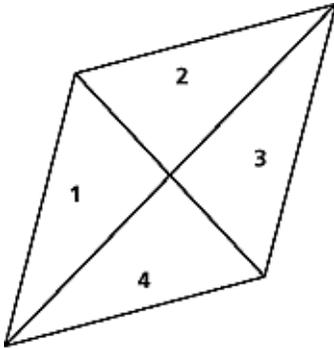
Slide the bottom triangle (number 4) straight up, aligning its bottom edge with the top edge of triangle 2. Slide the left triangle (number 1) to the right, aligning its left edge with the right edge of triangle 3.

[See Tip H2, page 119]

Homework, cont'd.

Here are some new cutting problems. When you find a process that you think works, justify the steps to be sure.

Problem H3. Start with a scalene triangle. Find a way to dissect it into pieces that you can rearrange to form a new triangle, but with three different angles. That is, no angles of the new triangle have the same measure as any angle of the original triangle. You should be able to demonstrate the following:

    a.   The final figure is a triangle. (It has exactly three sides.)

    b.   The two joined edges match.

    c.   All three angles are different from those of the original triangle.

   d.   As a challenge, can you solve this with just one cut?

[See Tip H3, page 119]

Problem H4. Start with a scalene triangle. Find a way to dissect it into pieces that you can rearrange to form a new triangle, but with three different sides. That is, no sides of the new triangle have the same length as any side of the original triangle. You should be able to demonstrate the following:

    a.   The final figure is a triangle. (It has exactly three sides.)

    b.   The two joined edges match.

    c.   All three sides are different from those of the original triangle.

Take It Further

Problem H5. Start with an arbitrary quadrilateral. Find a way to dissect it into pieces that you can rearrange to form a rectangle. Test your method on quadrilaterals like these. Try to justify why it will always work.
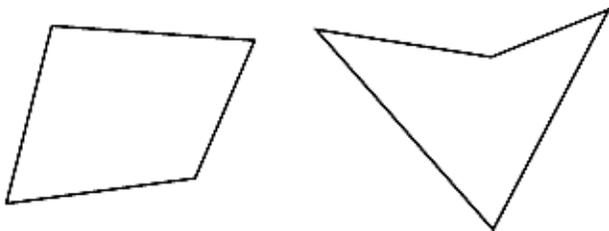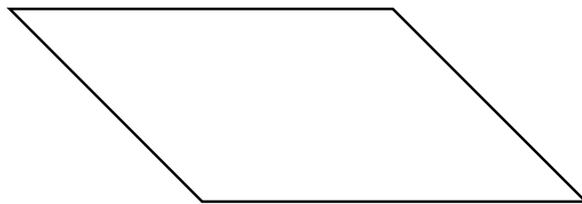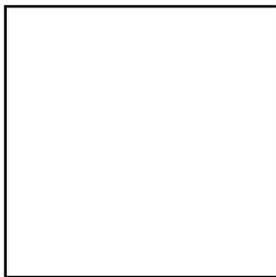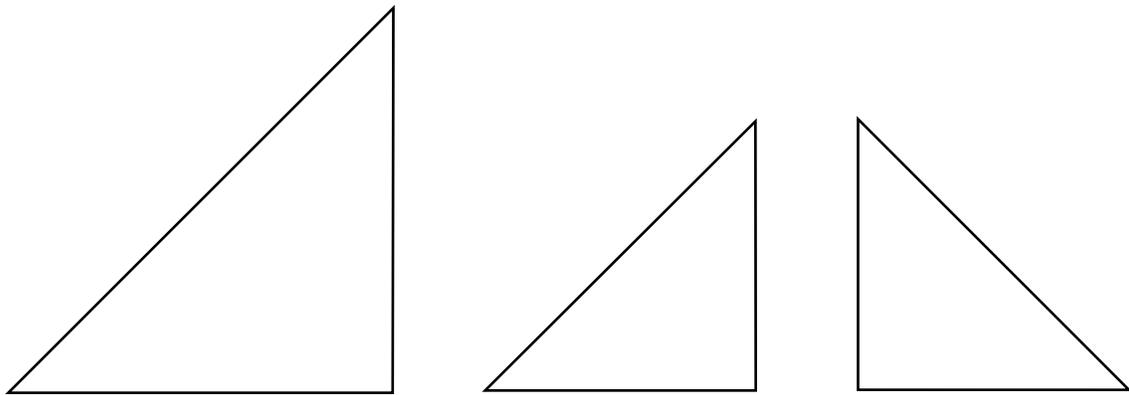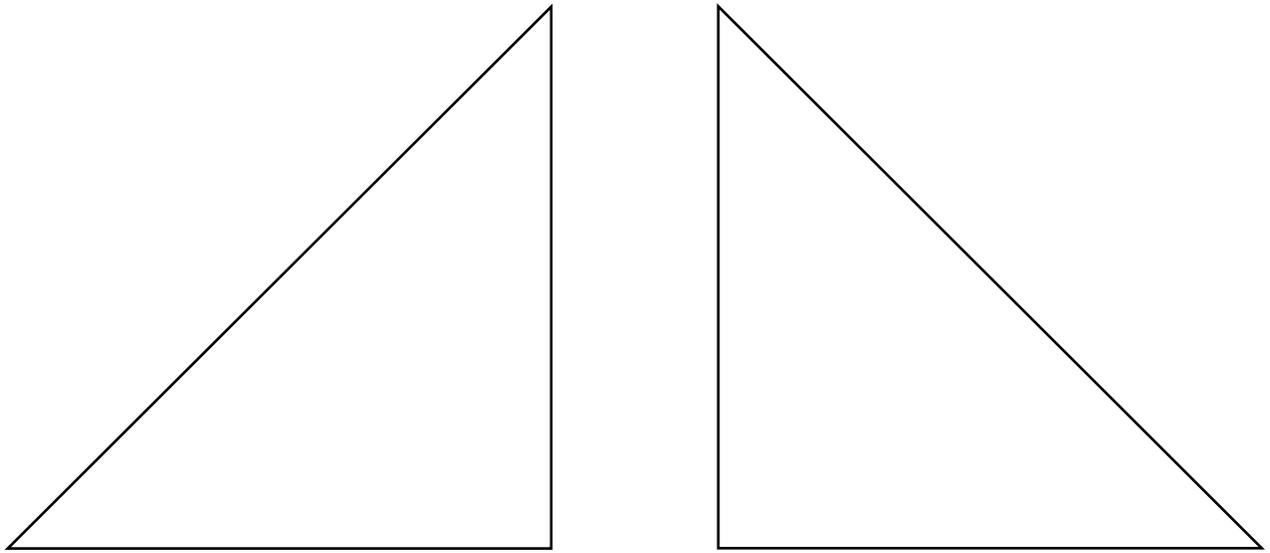

[See Tip H5, page 119]

Suggested Reading

This reading is available as a downloadable PDF file on the *Geometry* Web site. Go to www.learner.org/learningmath.

    Senechal, Marjorie (1990). Shape. In *On the Shoulders of Giants: New Approaches to Numeracy.* Edited by Lynn Arthur Steen (pp. 148-160). Washington, D.C.: National Academy Press.
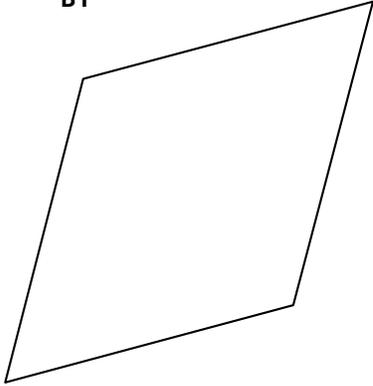
Tangrams

Cutting Up

B1

B2
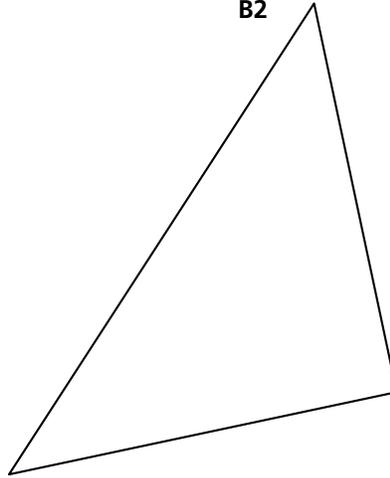
B3
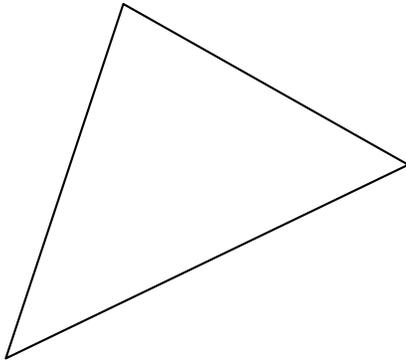
B4

B5

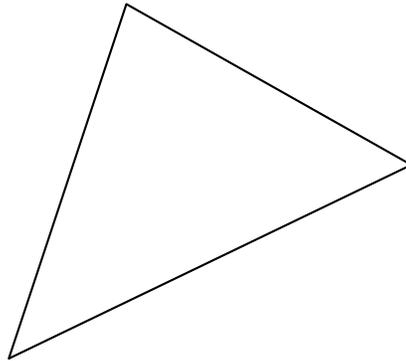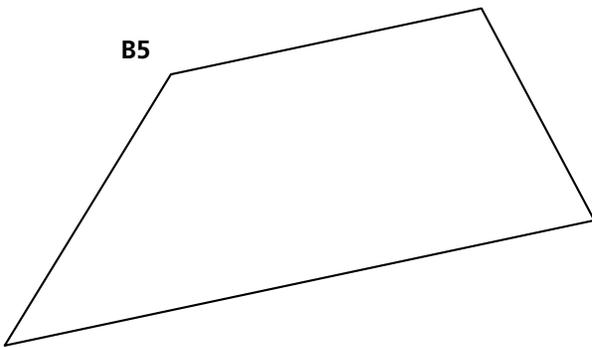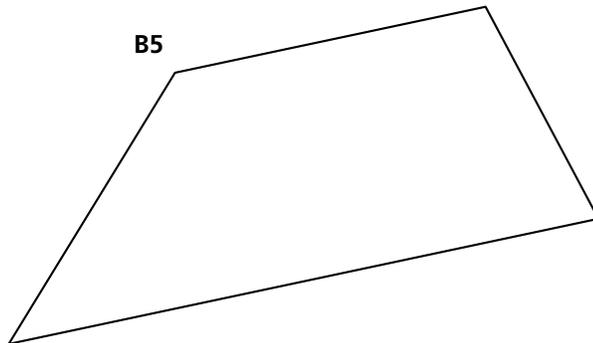**B5**

About Segment Notation

In mathematics, we make a distinction between an object and a measurement taken on that object. In the tradition of mathematicians being careful with their words and notation, this distinction carries over into how we write the names of things. For example:

$\overline{XY}$ refers to the segment with endpoints X and Y. The line over the letters tells you that we're talking about a segment, not a line or a ray.

XY, on the other hand, refers to the length of the segment with endpoints X and Y. The lack of segment (or line or ray) markings above the letters changes the meaning. We're no longer talking about the object itself, but about a property of that object (namely, its length).

This distinction continues when comparing segments. We say that two segments are congruent, so we use the segment notation:

$\overline{XY} \cong \overline{AB}$

But we say that two lengths are equal, so we use the length notation:

XY = AB

or

XY = 3 inches

The statements $\overline{XY} \cong \overline{AB}$ and *XY = AB* convey exactly the same information, but one is talking about segments and the other about lengths.

We make a similar distinction between angles and their measures. Thus, angles are congruent:

$\angle XYZ \cong \angle ABC$

But their measures are equal:

m∠XYZ = m∠ABC

or

m∠XYZ = 60°

Tips

Part A: Tangrams

Tip A1. Think about the angles you need to create in order for a shape to be a square.

Part B: Cutting Up

Tip B4. If you've solved Problems B1 and B3, you can put those solutions together to solve Problem B4. Alternately, you can divide the triangle into two right triangles and apply your solution to Problem B2 twice.

Part C: The Midline Theorem

Tip C3. Look at the angles marked 1 and 2 in the picture.

Tip C6. Consider the diagonal AC and its relationship to the segments in question.

Homework

Tip H2. Try this with a very long and skinny parallelogram.
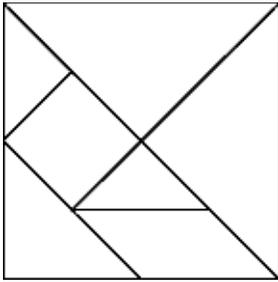
Tip H3. Form two angles that are smaller than the smallest one in the original triangle, and a third angle that is larger than the largest angle in the original.

Tip H5. Remember from Session 3 that you can divide any quadrilateral into two triangles.
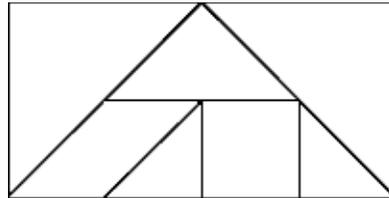
Solutions

Part A: Tangrams

Problem A1.

**Problem A2.** Here is one possible solution:

Problem A3.

a.   Translate the leftmost triangle horizontally to the right by twice the length of the side of the square.

b.   Rotate the top triangle 180° about vertex A. Then translate the square to the left by twice its side length.

c.   Imagine a line through the middle of the parallelogram, parallel to its horizontal sides. Reflect the rightmost triangle over it.

Part B: Cutting Up

Problem B1. Draw a perpendicular DE. Cut along the perpendicular to form a right triangle. Then translate the right triangle to the right until the side AD coincides with the side BC. Note that you could draw a perpendicular line anywhere along the side of the parallelogram (not just the vertex) as long as the perpendicular lies within the parallelogram. It is always possible to find one perpendicular line within the parallelogram between at least one pair of parallel sides.
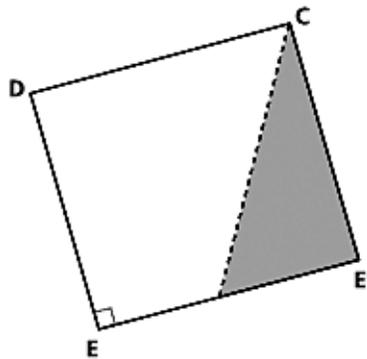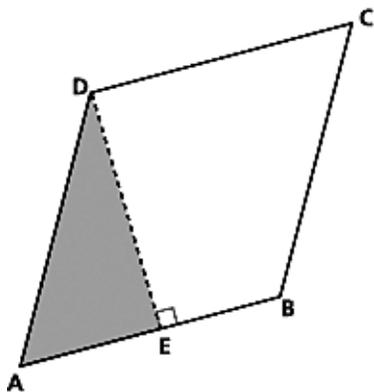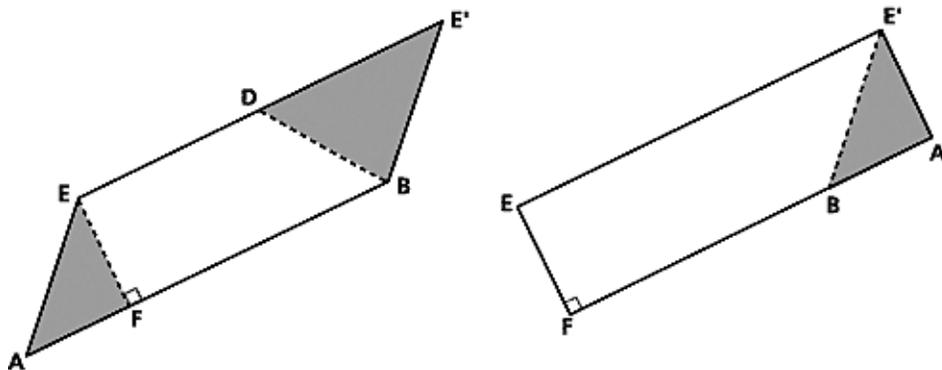
Solutions, cont'd.

Problem B2. Construct the midpoints of the sides BC and AC, namely D and E, respectively. Connect the two midpoints. Cut along the segment ED. Then rotate the triangle EDC 180° about the vertex E. Notice the sides EC and AE will coincide.
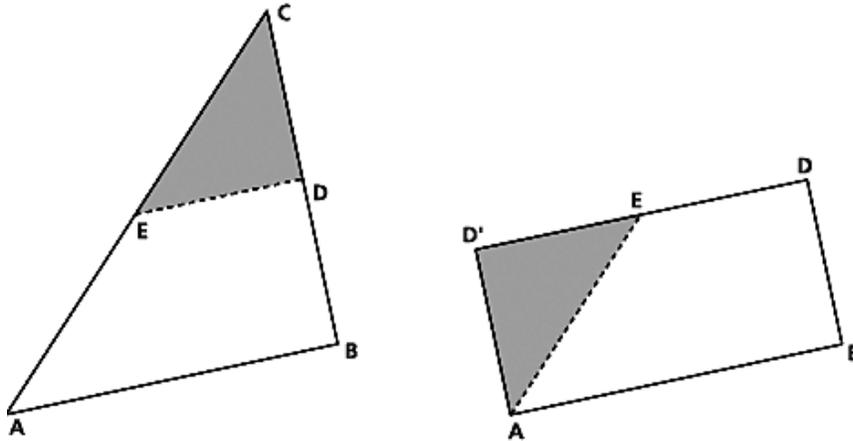
Problem B3. Proceed exactly as in Problem B2 by constructing a midline and rotating the top triangle 180° about the vertex D. The result will be a parallelogram instead of a rectangle.
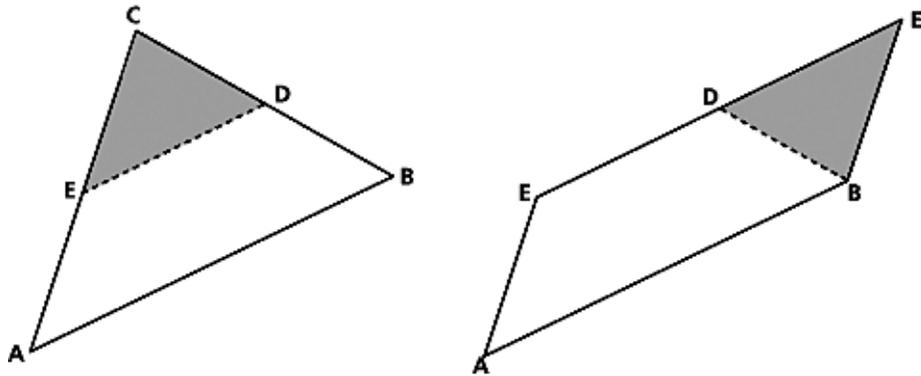
Problem B4. Start with a scalene, non-right triangle. Use the method of Problem B3 to form a parallelogram. Then apply the method of Problem B1 to get a rectangle from the parallelogram. Note that sometimes you may need to reposition the parallelogram before you turn it into a rectangle.
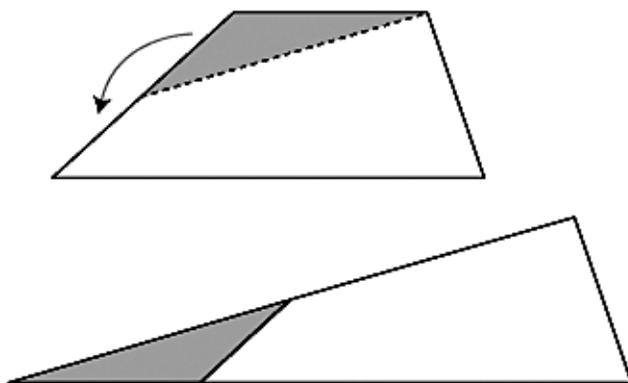
Solutions, cont'd.

Problem B5. Connect the midpoints G and H of the sides AD and BC, respectively, with a line segment GH. Cut along GH and rotate the trapezoid DCHG about the point G, counterclockwise, until the segments GD and AG overlap. The resulting figure will be a parallelogram. Then apply Problem B1 to create a rectangle.
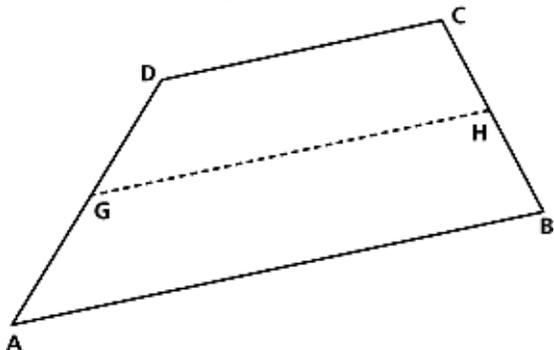
Problem B6. Starting with a trapezoid, you can use the process in Problem B5 to make a rectangle. Cut the rectangle into halves along its longer side (or shorter side, which works just as well). Then cut one of the smaller rectangles into two triangles by drawing a diagonal (see picture). Rotate triangle ADF clockwise about the vertex F until the sides DF and FC overlap. The resulting figure is a triangle.
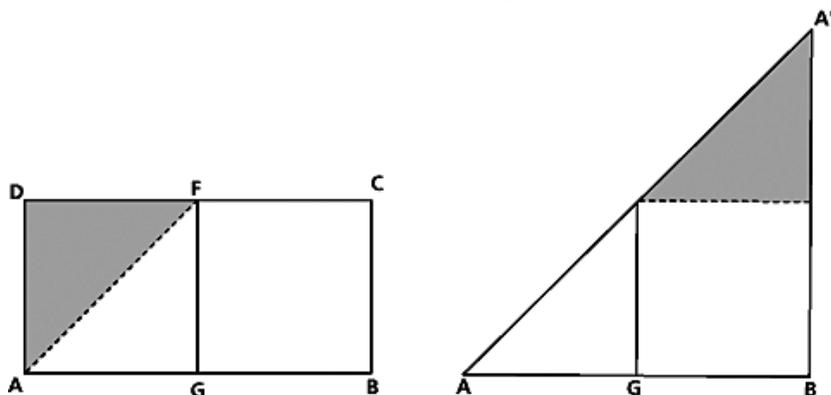
Challenge: Position the trapezoid so the parallel sides are horizontal, with the shorter one on top. Connect the top right vertex to the midpoint of the left (non-parallel) side. Cut along this segment to form a triangle on top and a quadrilateral. Rotate the triangle 180° about the midpoint of the left side. You now have a triangle with base = (sum of two bases of the trapezoid) and height that is the same as the height of the trapezoid.


The attached sides match up because you cut at a midpoint. The bottom side is straight because the bases are parallel in a trapezoid, so adjacent angles (bottom and top) are supplementary.

Solutions, cont'd.

Problem B7.

    a. Cut the rectangle along a diagonal, creating two triangles. By translating and flipping one of the triangles, put them together into one triangle whose congruent sides correspond to the diagonal of the original rectangle.

    b. Start with the rectangle ABCD. Find the midpoint, E, of one of its sides. Connect E with the vertex B, and cut along the segment EB. Rotate the triangle EBC counter-clockwise until the sides ED and CE overlap. The resulting triangle is a right triangle (see picture).
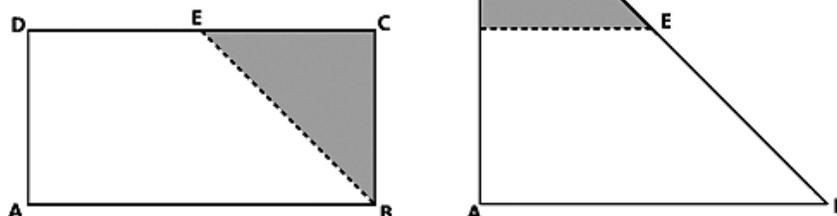

    c. Start with the rectangle ABCD. Mark an arbitrary point E (not the midpoint!) along the top side CD. Then find points M and N such that M is the midpoint of CE and N is the midpoint of DE.

        Cut along segment BM, and rotate triangle BCM 180° about point M. Segments EM and CM will coincide. Now cut along segment AN, and rotate triangle ADN 180° about point N. Segments DN and EN will coincide. Segments AD and BC will also coincide. (It's possible that one of the resulting sides will be the same as the side of the rectangle, but for all but three choices of point E—the excluded midpoint being one of the choices—this will produce a scalene triangle.)
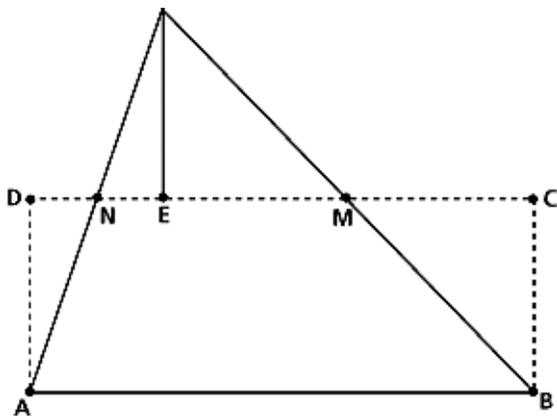

    d. Start with the rectangle ABCD. Choose any point E on the segment CD (other than C and D themselves). Connect E and B with a line segment. Translate the triangle EBC so that the sides BC and AD overlap. The resulting figure is a parallelogram but not a rectangle.
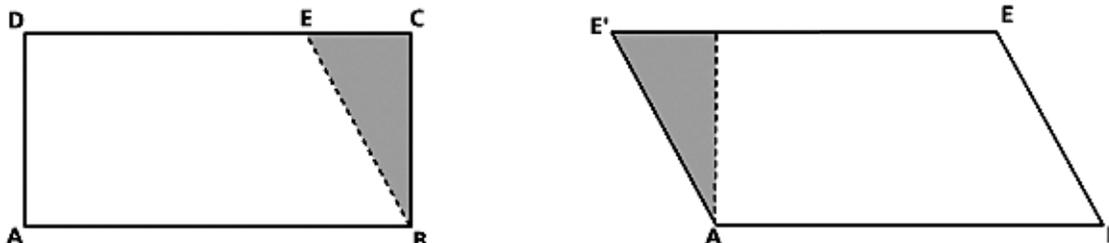
Solutions, cont'd.

Problem B7, cont'd.

   e. Use the solution from Problem B7 (d) to create a parallelogram. Take the shaded triangle and reflect it about a line parallel to the horizontal sides of the parallelogram, cutting it in half.
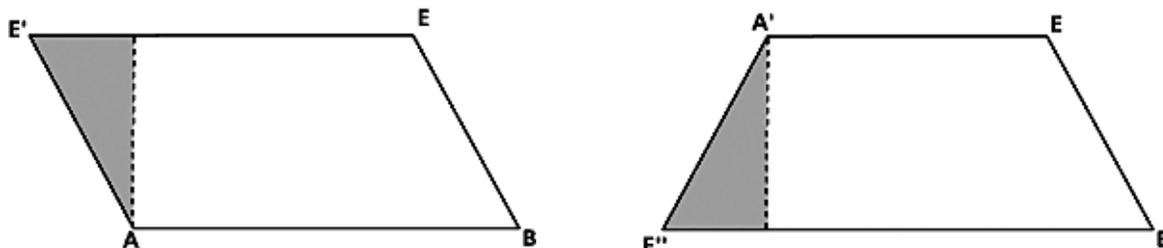
Part C: The Midline Theorem

Problem C1. By construction, E is a midpoint of BC, so the sides CE and EB are congruent. Similarly, the sides DE and EF are congruent. By Fact 1, the angles between the congruent pairs of sides are equal, since they are vertical angles (the two angles at the shared vertex E). Then, by Fact 2, the two triangles are congruent.

Problem C2. Since the triangles DEC and FEB are congruent (by Problem C1), the corresponding sides in the two triangles are congruent. In particular, DC and FB have the same length. But DC has the same length as AD, since, by construction, D is the midpoint of AC. Therefore, AD and BF have the same length.

Problem C3. The angles DCE and FBE have the same measure since they are corresponding angles in congruent triangles. If we extend line segments DC and FB into lines, the two angles become alternate interior angles along a transversal to the two lines. Since the alternate interior angles are congruent if and only if the two lines are parallel, we conclude that the two lines and the segments on them are indeed parallel. So we have shown that the segments AD and BF are parallel.

Problem C4. By Problems C2 and C3, we know that the segments AD and BF have the same length and are parallel. They are also opposite sides in the quadrilateral ABFD. Since we have one pair of opposite sides congruent and parallel, we can use the fourth definition in Fact 3 to conclude that ABFD is a parallelogram.

Problem C5. Since AB and DF are opposite sides of a parallelogram (by Problem C4), they are parallel. We also know that DE lies on DF since the points D, E, and F all lie on the same line (by construction). Therefore, DE is also parallel to AB.

Finally, since AB and DF are opposite sides of a parallelogram (by problem C4), they must have the same length. Also, DE and EF have the same length by our construction. This means that DE must be half as long as DF (which is DE + EF). Since DE is half as long as DF, it is also half as long as AB.

Solutions, cont'd.

Problem C6. Since EF is a midline of ABC, EF is parallel to AC and half the length of AC. Similarly, since HG is a midline of ACD, HG is parallel to AC and half its length. So EF and HG are parallel (since they are both parallel to AC) and of equal length (since they are both half the length of AC). The quadrilateral EFGH is a parallelogram, by Fact 3, since it has a pair of congruent and parallel opposite sides, namely EF and HG.

Thinking even more deeply, you might notice that ABCD was constructed to be a kite, so its diagonals are perpendicular. Therefore, the sides of the inside quadrilateral—each parallel to one of the diagonals of ABCD—are also perpendicular to each other. So, in fact, EFGH is a rectangle (a special kind of parallelogram).
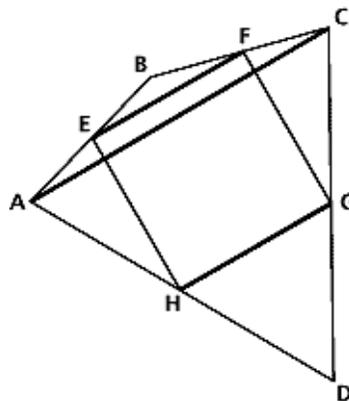
Problem C7. Using similar reasoning from Problem C6, you can prove that the inscribed quadrilateral must always be a parallelogram. In the diagram at right, construct the diagonal BD. Using this diagonal as the base of two triangles (BDC and BDA), we have two triangles with midlines: FG is the midline of triangle BDC, and EH is the midline of triangle BDA. Each midline is parallel to the base, so FG and EH must be parallel to one another. By a similar argument (using diagonal AC), we can say that EF and HG must each be parallel to AC, and therefore parallel to one another. Then, definition 1 in Fact 3 tells us that EFGH must be a parallelogram, no matter how the original quadrilateral is constructed.
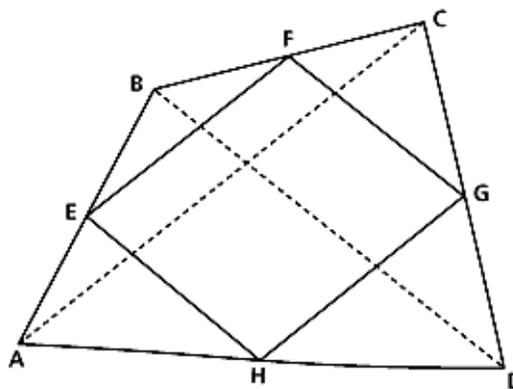
Note that for some concave quadrilaterals we can get the sides of the interior quadrilateral to coincide, so it's not really a parallelogram. Except in those rare cases, it will be a parallelogram.

Homework

Problem H1. Follow the instructions.

Step 1: The two shapes are congruent isosceles right triangles since all three pairs of corresponding sides are congruent, and two of the square's right angles are preserved, one in each triangle.

Step 2: Again, we get two congruent isosceles right triangles since the two legs of each are half the length of the original diagonal of the square, and since the angle between the two legs is the right angle (SAS congruence).

Step 3: We get an isosceles right triangle and a trapezoid. The base of the triangle is half the length of the longer base of the trapezoid (midline theorem), and is of same length as the shorter base of the trapezoid.

Step 4: The two shapes are congruent right trapezoids, each one of which has two right angles, one angle of 45° and one of 135°.

Step 5: We get a square and an isosceles right triangle and a square. These shapes are similar to those encountered in previous steps; i.e., their corresponding sides are proportional.

Solutions, cont'd.
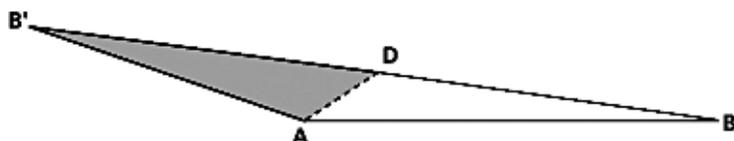
Problem H1, cont'd.

Step 6: The final two shapes are a parallelogram and an isosceles right triangle. The triangle is congruent to the one created in Step 5. The parallelogram has one pair of sides congruent to the leg of the triangle created, and one pair of sides congruent to the hypotenuse of the triangle just created.

Problem H2. The construction does not work in general. To see this, consider triangle number 4: In the final arrangement, one of its angles is also an angle of the final shape, which is supposed to be a rectangle. The angle in question is formed by the intersecting diagonals, so the construction only works if the diagonals of the original parallelogram intersect at an angle of 90°. This happens when the original parallelogram is a rhombus.

Problem H3. Start with a scalene triangle ABC. Find the midpoint D of the side opposite the smallest angle.
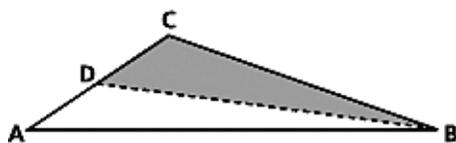

Cut along the segment DB, then rotate the triangle DCB about the vertex D by 180° (see picture). The final figure has three sides. Sides AD and DC match since they are of equal length, and points B, D, and B' are collinear since B'D is obtained by rotating DB by 180°. The angle at A is larger than any of the angles in the original triangle (since it is the sum of the two largest angles of the original triangle). The angles at B and B' are smaller than (hence different from) any of the angles in the original triangle since they are both smaller than the smallest angle in the original triangle. We only used one cut.

**Problem H4.** Let's say AB is the longest side of the original triangle.
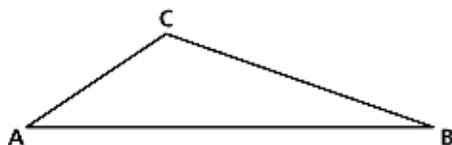
Draw the altitude CD (from vertex C to the side AB). Notice that it is shorter than any of the three sides of the original triangle.
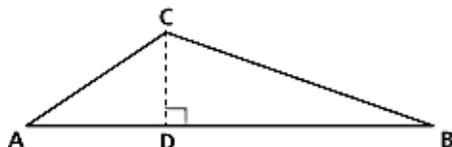
Solutions, cont'd.

Problem H4, cont'd.

Make the triangle into a rectangle using one of the methods we found in the session. Use side AB for the base. The other side will have length 1/2 CD (half the altitude).
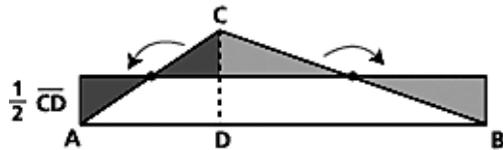
Now, cut along a diagonal of the rectangle. (Note that the diagonal is longer than side AB, and so it is longer than all three sides of the original triangle, since AB was longest.) This creates two small right triangles; call them FAB and BEF (they share two vertices; the right angles are at A and E).

Form an isosceles triangle by flipping and translating triangle BEF so that angles A and E (the right angles) are adjacent and sides FA and EB align, thus forming a new shortest side of the new triangle.
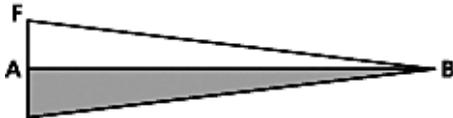

The shortest side of this triangle is twice the smaller side of our rectangle. That means it's the same as the altitude of the original triangle, and hence smaller than any of the original sides. The other two sides are the same, and since they are diagonals of the rectangle, they are longer than any of the original three sides.

Solutions, cont'd.

Problem H5. Split the quadrilateral in half along an interior diagonal. You have two triangles that have a side in common. For each triangle, use the common side as the base, and turn it into a rectangle using your algorithm from Problem B4. Now stack the two rectangles on top of each other. They match up because they have a common base (the diagonal of the original quadrilateral).

Quadrilateral 1:
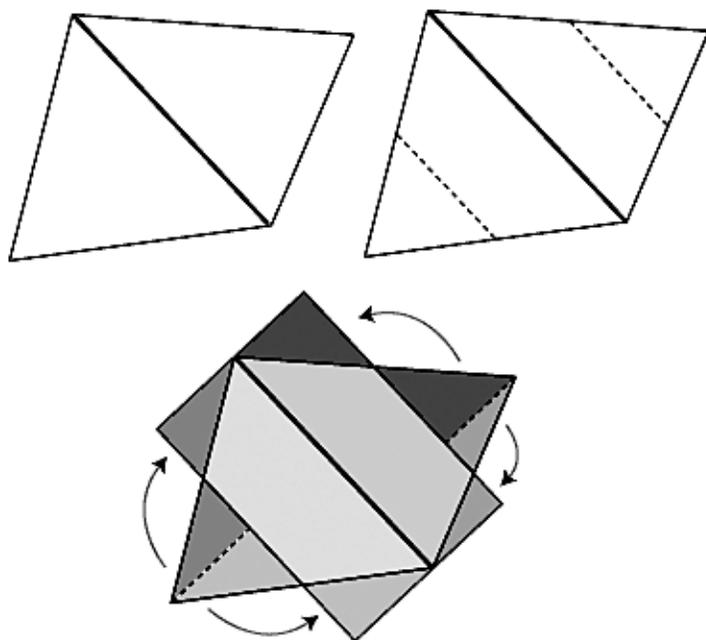
Quadrilateral 2:
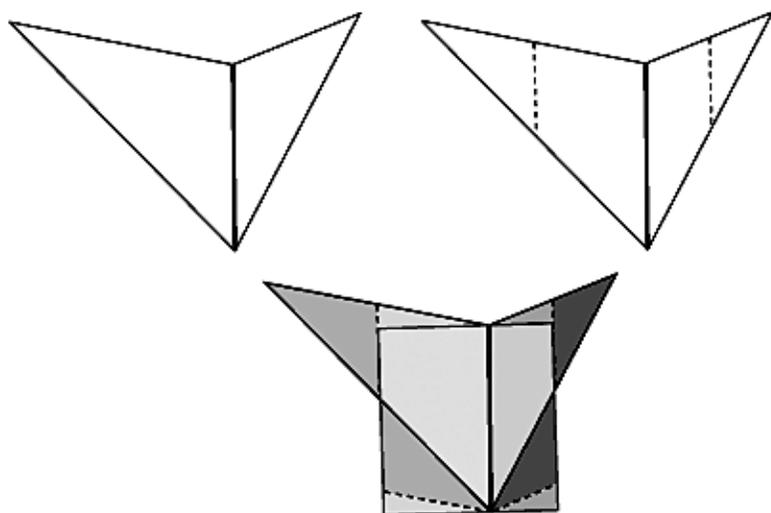
Note that in quadrilateral 2, there is only one interior diagonal to choose from, while quadrilateral 1 has two interior diagonals.