Session 10

Classroom Case Studies, Grades 3–5

This is the final session of the *Geometry* course! In this session, we will examine how geometry as a problem-solving process might look when applied to situations in your own classroom. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 3–5 begins below. Watch video program 10, for K–5 teachers, in this session. Go to page 215 for grades K–2 and page 247 for grades 6–8.

Key Terms for This Session

Previously Introduced

- congruent
- quadrilateral
- rectangle
- regular polygon
- similar
- square
- triangle inequality
- Venn diagram
- vertex

New in This Session

- van Hiele levels

Introduction

In the previous sessions, we explored geometry as a problem-solving process. You put yourself in the position of a mathematics learner, both to analyze your individual approach to solving problems and to get some insights into your own conception of geometric reasoning. It may have been difficult to separate your thinking as a mathematics learner from your thinking as a mathematics teacher. Not surprisingly, this is often the case! In this session, however, we will shift the focus to your own classroom and to the approaches your students might take to mathematical tasks involving geometry.

As in other sessions, you will be prompted to view short video segments throughout the session; you may also choose to watch the full-length video for this session. **[See Note 1]**

Learning Objectives

In this session, you will do the following:

- Explore the development of geometric reasoning at your grade level, including the van Hiele model of geometric learning
- Review mathematical tasks and their connection to the mathematical themes in the course
- Examine children's understanding of geometric concepts

Note 1. This session uses classroom case studies to examine how children in grades 3-5 think about and work with geometry. If possible, work on this session with another teacher or a group of teachers. A group discussion will allow you to use your own classroom and the classrooms of fellow teachers as case studies to make additional observations.

Part A: Geometry as a Problem-Solving Process (25 min.)

The study of geometry can include both problem solving and connections to other areas of mathematics (arithmetic, algebra, etc.). Too often, classrooms focus almost exclusively on correctly identifying shapes and their properties by name. While mathematical language and clear communication are important in geometry, it is important to include other kinds of geometric problems as well so that geometry isn't reduced to mere nomenclature. **[See Note 2]**

When viewing the following video segment, keep the following questions in mind:

a. How does the teacher incorporate geometric language into the lesson without making it the focus of the lesson?

b. Where in the lesson are students learning new geometric content? What is that content?

c. Where in the lesson are students solving problems and thinking mathematically? How does the problem solving relate to the geometric content?

d. Thinking back to the big ideas of this course, what are some geometric ideas these students are likely to encounter through their investigation of this situation?

Video Segment (approximate time: 9:38-12:01)**:** You can find this segment on the session video approximately 9 minutes and 38 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, fifth-grade students in Ms. Kurchian's class are working to sort different figures into Venn diagrams. They discuss whether shapes fit certain criteria given by the labels on their diagrams and when shapes can fit multiple criteria.

Problem A1. Answer the questions you reflected on as you watched the video:

a. How does the teacher incorporate geometric language into the lesson without making it the focus of the lesson?

b. Where in the lesson are students learning new geometric content? What is that content?

c. Where in the lesson are students solving problems and thinking mathematically? How does the problem solving relate to the geometric content?

d. Thinking back to the big ideas of this course, what are some geometric ideas these students are likely to encounter through their investigation of this situation?

Note 2. Before examining specific problems at this grade level, you will watch with an eye toward geometric problem solving a teacher who has also taken the course in her classroom. The purpose in viewing the video is not to reflect on the teacher's methods or teaching style, but to watch closely the way she brings out geometric ideas while engaging her students in a problem-solving task.

Part A, cont'd.

Problem A2. This lesson is not couched in a "real-world context." Students are sorting shapes and thinking about mathematical ideas in the abstract. What are the advantages and disadvantages of this kind of lesson? Are "mathematics only" lessons important in your classroom? What purpose do they serve, as opposed to contextualized lessons? **[See Note 3]**

Join the Discussion **www.learner.org**

Post your answer to Problem A2 on an email discussion list; then read and respond to answers posted by others. Go to the *Geometry* Web site at www.learner.org/learningmath and find Channel Talk.

Problem A3. Ms. Kurchian's lesson was based on a lesson from Session 3, Part B (the Polygon Classification Game). Discuss the ways Ms. Kurchian's lesson was similar to and different from the one in this course. What adaptations did she make and why?

Note 3. This is a particularly good discussion to have with your colleagues. Everyone has different opinions and thoughts about the use of context in the mathematics classroom. Spend some time talking about not just what you think, but why you think it. Cite examples from your own experience instead of focusing on what you have heard others say.

Part B: Developing Geometric Reasoning (40 min.)

Introducing van Hiele Levels

The National Council of Teachers of Mathematics (NCTM, 2000) identifies geometry as a strand in its *Principles and Standards for School Mathematics*.* In grades pre-K through 12, instructional programs should enable all students to do the following:

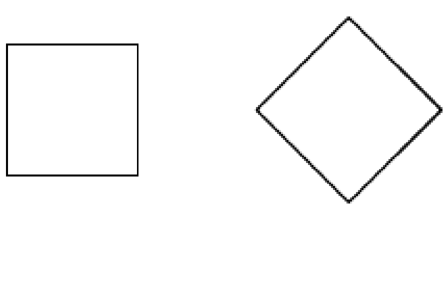
- Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Apply transformations and use symmetry to analyze mathematical situations
- Use visualization, spatial reasoning, and geometric modeling to solve problems

In grades 3-5 classrooms, students are expected to do the following:

- Identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes
- Classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids
- Build and draw geometric objects
- Create and describe mental images of objects, patterns, and paths
- Use geometric models to solve problems in other areas of mathematics, such as number and measurement
- Recognize geometric ideas and relationships and apply them to other disciplines and to problems that arise in the classroom or in everyday life

Dutch educators Pierre van Hiele and Dina van Hiele-Geldof developed a theory of five levels of geometric thought. It is just a theory, but a useful one for thinking about activities that are appropriate for your students and prepare them to move to the next level, and for designing activities for students who may be at different levels.

Level 0: Visualization. The objects of thought at level 0 are shapes and what they look like. Students have an overall impression of the visual characteristics of a shape, but are not explicit in their thinking. The appearance of the shape is what's important. Students may think that a rotated square is a "diamond" and not a "square" because it looks different from their visual image of square. *(Early elementary school and, for some, late elementary school)*



* *Principles and Standards for School Mathematics* (Reston, VA: National Council of Teachers of Mathematics, 2000). Standards on Geometry: Grades 3-5, 41, 164. Reproduced with permission from the publisher.

Part B, cont'd.

Level 1: Analysis. The objects of thought here are "classes" of shapes rather than individual shapes. Students are able to think about, for example, what makes a rectangle a rectangle. What are the defining characteristics? They can separate that from irrelevant information like the size and the orientation. They begin to understand that if a shape belongs to a class like "square," it has all the properties of that class (perpendicular diagonals, congruent sides, right angles, lines of symmetry, etc.). *(Late elementary school and, for some, middle school)*

Level 2: Informal Deduction. The objects of thought here are the properties of shapes. Students begin "if-then" thinking; for example, "If it's a rectangle, then it has all right angles." Students can begin to think about the minimum information necessary to define figures; for example, a quadrilateral with four congruent sides and one right angle must be a square. Observations go beyond the properties into mathematical arguments about the properties. Students can engage in an intuitive level of "proof." *(Middle school and, for some, high school)*

Level 3: Deduction. The objects of thought here are the relationships among properties of geometric objects. Students can explore relationships, produce conjectures, and start to decide if the conjectures are true. The structure of axioms, definitions, theorems, etc., begins to develop. Students are able to work with abstract statements and draw conclusions based more on logic than intuition. *(This is the goal of most 10th-grade geometry courses, but many students are not developmentally ready for it.)*

Level 4: Rigor. The objects of thought are deductive axiomatic systems for geometry. For example, students may compare and contrast different axiomatic systems in geometry that produce our familiar Euclidean plane geometry, finite geometries, the geometry on the surface of a sphere, etc. **[See Note 4]**

For more information on the van Hiele levels and how to work with students within each level, read the article "Geometric Thinking and Geometric Concepts" by John A. Van de Walle from *Elementary and Middle School Mathematics*. This reading is available as downloadable PDF files on the *Geometry* Web site. Go to www.learner.org/learningmath.

Van de Walle, John A. (2001). Geometric Thinking and Geometric Concepts. In *Elementary and Middle School Mathematics: Teaching Developmentally, 4th ed.* (pp. 342-349). Boston: Allyn & Bacon.

Note 4. If you are working with a group of colleagues, take some time to discuss your own students. Where in the van Hiele levels do you see them functioning comfortably? (There will be a range, of course, because not all students are the same.) Try to cite evidence from your classrooms: With which tasks do students find success? With which tasks do they struggle?

Part B, cont'd.

Analyzing With van Hiele Levels

In this course, we have primarily worked across levels 2-4. You may feel that the activities we've done are not appropriate for the level of your students, and you're probably right. The goal for this session is for you to think about problems and activities that are at your students' level, and how to help them prepare for the next level of thinking.

In grades 3-5, students should be moving from working easily at level 0 through level 1. By fifth grade, they should be starting to work at level 2, thinking in more sophisticated ways.

Video Segment (approximate time: 14:03-20:24)**:** You can find this segment on the session video approximately 14 minutes and 3 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch this clip from Ms. Kurchian's class, and think about how both the lesson and the teacher are encouraging students to move to that next level of geometric reasoning. **[See Note 5]**

Problem B1. Where in the video do you see evidence of the following?

- (Level 1 thinking) Students thinking about classes of shapes rather than the individual shapes in front of them. Do students seem concerned with orientation or size of the figures they use compared with the ones on the board?
- (Level 2 thinking) "If-then" reasoning and making geometric arguments

Problem B2. In Session 9, Part A, you worked on the problem of building the five Platonic solids and then arguing from the construction that only five such solids were possible. Recall your own experience in this activity as an adult mathematics learner. During the activity, when did you have to use level 2 thinking? (How did you know when to stop building with triangles and move on to other figures? How did you convince yourself that no other Platonic solids were possible?)

Problem B3.

a. What do you think were the key pieces of geometry content in this activity? What knowledge did you learn, solidify, or connect with better?

b. What do you think were the key thinking and reasoning skills in this activity? How did the reasoning and geometric content tie together?

Problem B4. Now think about students in grades 3-5 and how the Platonic solids activity might work with them. What must students know and be comfortable with to get the most out of this activity? What are potential stumbling blocks for them?

Join the Discussion — www.learner.org

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Problem B5. What might students misunderstand or find confusing in the lesson? How could you alter the lesson or prepare them beforehand to help avoid these misunderstandings?

Note 5. Again, remember that the focus of the video case study is not to examine teaching practice, but to focus on the students and their thought processes.

Part C: Problems That Illustrate Geometric Reasoning (55 min.)

Geometric Reasoning Problems, Part 1

In this part, you'll look at several problems that are appropriate for students in grades 3-5. As you look at the problems, answer these questions:

a. What is the geometry content in this problem?

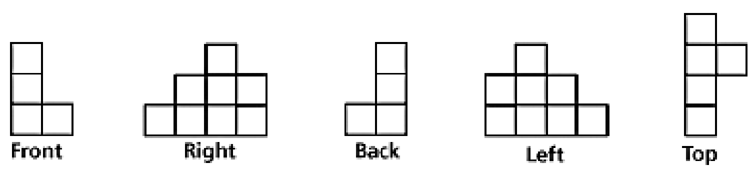
b. What skills do students need to work through this problem? What skills will this problem help them develop for later work?

c. What level of geometric thinking is expected of students in the problem? Does it ask students to bridge levels?

d. What other questions might extend students' thinking about the problem?

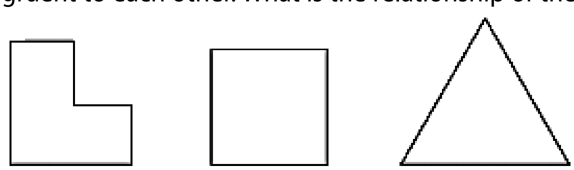
e. Describe a lesson that you could develop based on the content of this problem. **[See Note 6]**

Problem C1. Given the front, right, back, left, and top views shown here, use cubes to build a building that fits the pictures.



[See Note 7]

Problem C2. Start with each shape shown below. Find a way to cut the shape into four smaller shapes, all congruent to each other. What is the relationship of the smaller shapes to the larger one?

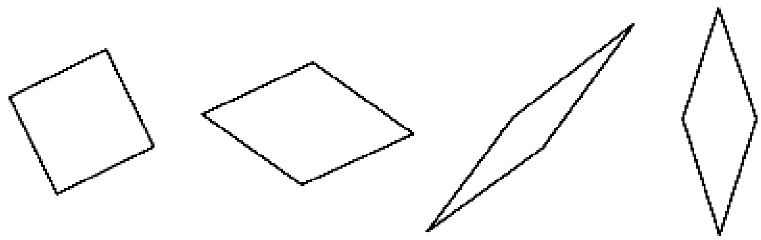


Note 6. It's difficult to identify the important content and how students might approach an activity without actually doing the mathematics yourself. These are, for the most part, short problems and activities. Allow yourself time to work through the mathematics, even briefly, before going on to answering the other questions.

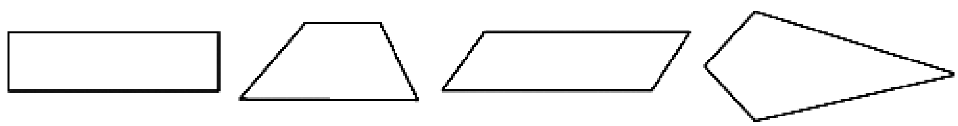
Note 7. Additional questions you might explore (or ask students to): What is the relationship between the right and left views? Front and back views? Will that always happen, or is there something special about this building? If you were to provide the minimum information for someone to copy the building, what would you choose? (Think about the similarity between this and questions about the minimum information necessary to determine if two triangles are congruent.) Can you build two different buildings with the same set of five views? (And how do you decide if two buildings are different?)

Part C, cont'd.

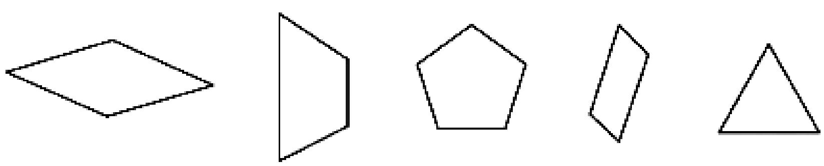
Problem C3. All of these figures have something in common.



None of these has it.



Which of these has it?



Geometric Reasoning Problems, Part 2

As you look at the next set of problems, answer these questions:

a. What is the geometry content in this problem?

b. What skills do students need to work through this problem? What skills will this problem help them develop for later work?

c. What level of geometric thinking is expected of students in the problem? Does it ask students to bridge levels?

d. What other questions might extend students' thinking about the problem?

e. Describe a lesson that you could develop based on the content of this problem.

Problem C4. Use any materials that allow you to construct the following shapes. Some may not be possible. If you think one of the constructions is impossible, say what makes you think so.

a. Make a four-sided shape with two opposite sides the same length but not parallel.

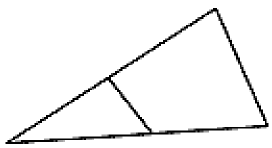
b. Make several six-sided shapes, some with one pair of sides parallel, some with two pairs of sides parallel, some with three pairs of sides parallel, and some with no pairs of sides parallel.

c. Make some shapes with all 90° angles. Make shapes with this property and three, four, five, six, and seven sides.

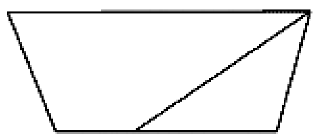
Part C, cont'd.

Problem C5. List all the properties of a rectangle that you can think of. From your list, choose sets of two or three statements that you think would make a good definition for a rectangle. Is there more than one possible definition? **[See Note 8]**

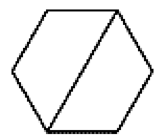
Problem C6. Begin with a convex polygon with a given number of sides. Connect two points on the figure to form two new polygons. What is the total number of sides in the two resulting polygons?



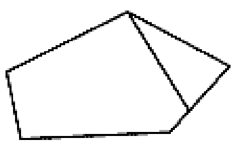
Begin with 3 sides
Two new shapes: 7 sides



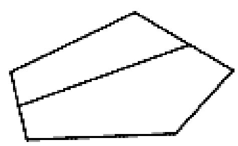
Begin with 4 sides
Two new shapes: 7 sides



Begin with 6 sides
Two new shapes: 8 sides



Begin with 5 sides
Two new shapes: 8 sides



Begin with 5 sides
Two new shapes: 9 sides

Note 8. You may want to review our work with creating and understanding definitions in Session 3, Part C.

Homework

Solutions are not provided for these homework problems, since answers will vary depending on individual experiences.

Problem H1. Interview a teacher in the grade level above you. Pick one of the problems in this session, and ask him or her the following questions:

a. How does the content of this problem prepare students for geometric thinking in your grade?

b. Why do you think this content is important?

c. How could this problem be extended for students in your grade?

Problem H2. Look at a lesson or activity in your own mathematics program for your grade level that you think has potential for developing students' geometric reasoning. If you were to use this lesson or activity now, after taking this course, how might you modify or extend it to bring out more of the important ideas about geometry?

Solutions

Part A: Geometry as a Problem-Solving Process

Problem A1.

a. Answers will vary. Some ideas: The lesson is really one of analysis, but students must know and use geometric vocabulary to work with the labels and to communicate with their group about the problems.

b. It may not be clear what from the lesson is new to the students, but it seems that some of the logical relationships, and possibly some of the vocabulary, might be.

c. Answers will vary. Throughout the lesson, students are thinking analytically to solve problems such as placing the polygons into appropriate Venn diagrams and answering questions about which properties are shared by certain groups of polygons and which ones are not.

d. One of the geometric ideas of the course that students encounter in this lesson is the idea of classification. In this lesson, we see how students extend their knowledge of classification of polygons by thinking about categories of shapes and analyzing them in terms of properties that they share. Classification also progresses through grade levels, and here we see how the class develops from the more basic ideas and vocabulary towards more complex levels of thinking and problem solving.

Problem A2. People have very different, and often very strong, opinions about the use of context in mathematics classrooms. It is important to present students with a variety of lessons. Students can be engaged by problems that are not context-based, as well as by those with real-world connections.

Problem A3. There were lots of adaptations—the labels and shapes were selected to be more familiar to students in the class. The students used just two loops in the Venn diagram rather than three loops. The activity is more carefully structured in terms of different groups working with different pairs of labels.

Part B: Developing Geometric Reasoning

Problem B1. Answers will vary. Some possible responses:

- (Level 1 thinking) In placing shapes into the circles, Stephanie and Cara don't worry about the orientation. They think of the particular shapes as representing all of the same type of figure. Also, they see the right angle at the vertex of the kite, even though it is oriented in an unfamiliar way, and appearing in an unfamiliar figure.
- (Level 2 thinking) Students in the class make guesses about the labels and refine their guesses based on additional information. If the pentagon goes in the right circle, the label can't be quadrilaterals.

Problem B2. Answers will vary. The thinking often goes like this: "If it's going to make a solid shape, there must be at least three polygons meeting at a vertex. If it's going to make a solid shape, then there must be less than a total of 360° around a vertex. Regular hexagons and polygons with more than six sides all have 120° or more at each vertex, so these shapes cannot be used." And so on.

Problem B3. Answers will vary. Some possible answers:

a. Key pieces of geometry are definitions and properties of regular two- and three-dimensional figures, building polyhedra, angle relationships, and so on. We also explored Euler's formula (see Session 9, Problem A8).

b. "If-then" thinking, reasoning through every possible case, and generalizing were all important parts of the activity. It was important to both know the geometry (what are the angle measures for polygons with different numbers of sides?) and use those facts in making deductions.

Solutions, cont'd.

Problem B4. Answers will vary. Students will probably gain understanding of three-dimensional figures and how they're different from polygons. They will likely gain valuable understanding and visualization skills from building and manipulating the solids, and from attempting to count faces, edges, and vertices. They may not have the prerequisite knowledge of angle measures in polygons as a solid foundation. Some students may also struggle with the generalizations. If six triangles don't work, how do we know seven triangles won't work? Why can we eliminate polygons with seven, eight, and more sides without even trying to build them?

Problem B5. Answers will vary. Some ideas: Lots of experience with building generalizations in cases that are easier to check, and lots of experience with polygons will help.

For example, students may experience some difficulty in thinking about the role of angles and their connection to building Platonic solids. To prepare them beforehand, you may want to work on the sum of the angles in a triangle and extend it onto making generalizations about other polygons through dividing them into triangles as you've seen in Session 3, Part A of this course.

Part C: Problems That Illustrate Geometric Reasoning

Problem C1.

a. This problem is mostly about visualization and relating three-dimensional objects to two-dimensional representations.

b. By building three-dimensional objects, students strengthen their spatial sense and their intuitive knowledge of shapes and their properties. Two-dimensional representations help their abstract thinking about geometric properties. It also helps students with visualization and reasoning about 3D figures, which are an important part of high school geometry. Visualization and going from 2D to 3D (and vice versa) were important parts of this course.

   There are no real prerequisites to working through this problem. Students should have some familiarity with the blocks as building materials and with the idea of two-dimensional drawings representing three-dimensional figures. This problem helps prepare students for later work, when they must reason about drawings, drawing conclusions about figures based on information in a sketch.

c. This problem represents level 0 (informal exploration of shapes and visualization) and level 1 thinking (the drawings of squares represent cubes in which relative position, not size, matters; this requires analyzing skills on the students' part). There could also be possible bridges to level 2 (using "if-then" thinking to help build the tower).

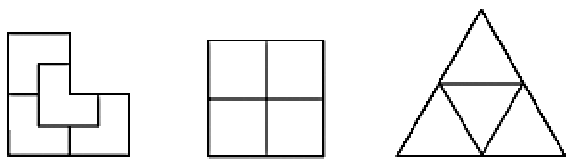
d. Ways to extend students' thinking include the following: Give a set of views that can create more than one possible structure (think of a "hole" somewhere in the middle that is undetectable by any of the views provided). Ask students to create different structures with the same front view. Ask students to eliminate extra information, realizing that the left and right views are mirror images, as are front and back, so really just three views provide the required information.

e. One possible lesson: Give groups of students sets of five views to build from. Have "answer cards" (with the structure pictured) ready for them to check their work. If they are successful, they can move to a more challenging piece. Move towards a game where students try to build the structures with the least number of views needed.

Solutions, cont'd.

Problem C2.

a. This problem deals with congruence and similarity, as well as (potentially) leading to thinking about measures like area and perimeter. Students are required to reason through different properties in order to examine which ones will ensure congruence of the smaller shapes. Students might also notice that the area and the perimeter of the smaller shapes are, in each case, equal. This may be helpful to students when finding the solution, but this is not enough to be the test for congruence. One possible solution follows:



Note that with the square, it could be cut into four congruent rectangles by making equally spaced cuts parallel to one pair of sides. This is certainly a solution to the problem posed, though the shapes in this case are not similar to the original.

Students will also be looking at the properties of the smaller shapes and comparing them to those of the original, larger shapes, and deciding what makes the two figures similar. They may notice, for example, that the four smaller triangles all have the same angles as the larger one, as well as the sides that are in proportion with the sides of the larger one.

b. The openness of the problem prepares students for thinking more creatively about geometric problems. The problem is similar to some of the cutting and dissection work from Session 5, Part B. This kind of problem prepares students for working more formally with ideas of congruence and similarity later. (If they have a solid grasp of the concept through informal activities, it is much easier to formalize these notions and build up to mathematical proof.) It also helps extend students' visualization and problem solving. With the first shape in particular, they will probably have to think creatively and try several different ideas before getting it right.

c. This represents level 1 thinking (seeing figures as congruent or similar, even if they are differently oriented).

d. To extend students' thinking, you might make a table comparing perimeters and areas of the smaller and larger figures. Several ideas should come up:

- If shapes are congruent, they must have the same perimeters and the same areas, so it's only necessary to check one.
- If four congruent shapes fit inside the larger, each must contribute 1/4 to the area. There's no need to measure here.
- If the four triangles really are congruent, for example, the sides must each be half as long as the original. (Two small sides make up a large side on each side of the triangle.) Again, no need to measure; if each side is 1/2 as long, the total perimeter must be 1/2 as much.

e. One possible lesson would involve starting with a review of the ideas of congruence and getting a definition from students (for example, two shapes that are identical, so you can fit one on top of the other, etc.). Next, hold up a standard sheet of paper and ask for several ways to cut that sheet into two congruent pieces. Follow through with each suggestion (cut parallel to the short sides through the midpoints, parallel to the long sides through the midpoint, and along a diagonal to form two congruent triangles). In each case, use the "fit on top of each other" test to make sure the result is two congruent figures. Give students equilateral and L-shaped figures, and ask them to cut those into two congruent pieces. Let them share their results. Finally, pose the challenge problem of cutting the figures as described, with students making a poster or display of their solution.

Solutions, cont'd.

Problem C3.

a. This problem requires building a definition from examples and then applying the definition to other cases. Here students will be working with the definition of equilateral polygons, or more specifically, quadrilaterals. (Polygons that have equal sides are called equilateral.) Next, they will look at quadrilaterals that do not have equal sides, and finally, they will extend the definition onto other polygons that may or may not share that property. For example, regular pentagons or regular triangles would share the property of having equal sides.

b. This requires focusing on properties of figures and reasoning about them, and in this way it is similar to the polygon classification in Venn diagrams from Session 3, Part B. Creating definitions and understanding definitions that have been created by others is an essential part of learning and doing mathematics. A key part to understanding a mathematical definition is to create both examples and non-examples for yourself. See Session 3, Part C for more on creating and understanding mathematical definitions. Activities like this encourage students to play with mathematical ideas and develop these important habits for later work.

c. This problem represents level 1 thinking (the focus is on properties rather than specifics of a figure; students must consider all the shapes in a class of shapes, rather than a single one) and some level 2 thinking (if this shape has the property, then it can't be this property; if I need to make decisions about triangles and pentagons, then the property can't depend on number of sides).

d. Ways to extend students' thinking include similar activities with more complicated (or less familiar) shared properties; asking students to give a name (either standard or invented) to the property and write a definition of it; and providing students with a written definition for which they create examples and non-examples.

e. One lesson might go like this: Choose an attribute that several but not all students in the class have in common; for example, wearing sneakers or having long hair. Select several but not all of the students with that attribute to come to the front of the room. Tell the class that their job is to be detectives; that all of these students have something in common, and their job is to figure it out. Next, bring up several students who do not fit the profile, and announce that all of those students do not have the attribute in question. Finally bring up five or so students, some with the attribute and some without. Ask students to silently write on their papers the names of students they think "belong" to the original group, and those who do not. Have the group return to their seats, where they can make their guesses as well. Finally, ask for a show of hands for who thought each student belonged, and why—what was the property in question? When this introductory activity is finished, tell students that they will now play detective with shapes. Pass out several worksheets like the one provided. You might add places where students can do the following:

   - Write down a name or description of the property they detected
   - Draw two other examples and non-examples of their own
   - Create their own detective worksheet to share with a partner

Problem C4.

a. This problem focuses on properties and thinking about classes of figures. Students also have the opportunity to develop their thinking about the relationship between the number of sides and angles within a shape. For example, in part (c) they will discover that the only shape possible, if all the angles are 90º, will be one with four sides. This is similar to other classification problems and activities, but differs in that the properties (or the combinations of properties) are less familiar, so they require more creative thinking on the part of students.

Solutions, cont'd.

Problem C4, cont'd.

b. To work through this lesson, students should already know all of the vocabulary. The challenge should be creating figures with the properties, not understanding what the properties are. This prepares students for thinking about some difficult mathematical ideas, like how you create examples to show something is possible and how you convince yourself and others that something is impossible (a much harder task). Both of these are key parts of mathematical proof. The first is an example of proof by construction (proving that something exists by creating it). The second has a long and famous history in mathematics, including impossibilities like creating a formula like the quadratic formula to solve 5th-degree polynomials (equations with an x^5 in them); trisecting a general angle with straightedge and compass; and "squaring the circle"—creating a square with the same area as a given circle, again using straightedge and compass.

c. This is a level 1-focused problem (it requires students to examine a particular property within an entire class of shapes). The addition of combinations of properties that are impossible, and asking students to justify how they know it is impossible, bridges this into a level 2 task as well.

d. To extend students' thinking, you can ask them to explain why some of the constructions are impossible—what about a pentagon prevents you from making one with five right angles? This is essentially creating a proof that the construction is impossible. You can also ask students to salvage the bad constructions—if you can't make a pentagon with five right angles, what's the best you can do? Can you make one with two right angles? Three? Four?

e. One possible lesson: Hand out the materials students will be using for construction. (Possibilities include toothpicks and gumdrops, linkage strips, or other construction materials with sticks and connectors.) Allow students a few minutes to explore the materials, then pose the first construction challenge to the whole class. Ask several students to hold up their solutions and explain why they meet the construction criteria. Pose a second construction challenge. (Make sure the first two use familiar vocabulary and are both possible to do.) Finally, pose one of the impossible challenges and give students several minutes to work on it. Eventually, someone will exclaim that it can't be done. Ask the class who thinks it is possible and just hard to do, and who thinks it is impossible. Ask students to explain their opinions and try to convince each other. When the class (with some help from you) understands that sometimes there's no shape that satisfies the requirements, hand out several more construction challenges. Tell them that some are possible and some are impossible. Their job is to solve the possible ones, and to draw one (or more) example of what they created. For the impossible ones, they should write that it's impossible, and try to explain what goes wrong.

Problem C5.

a. This task is focused on definitions and the equivalence of different sets of properties. For example, some properties listed may include the following:

   - Rectangles are quadrilaterals; they have four sides.
   - All angles must be 90º.
   - Opposite sides are parallel.
   - The sum of the angles is 360º.

   None of these alone is sufficient to define a rectangle; to do that, the statement must be true of every rectangle, and not true of anything that's not a rectangle. All of the statements above are true of every rectangle, but you can also create non-rectangular shapes that satisfy them.

b. To tackle this task, students must first be very comfortable with the shape in question (here it's a rectangle, but the activity is easily adapted to anything else). They should also have some experience with activities like the one in Problem C3, since in this case they will have to create their own examples and non-examples to construct an adequate definition. This activity prepares students for more formal mathematical reasoning, both in thinking about definitions and also in proof, where careful (and flexible) use of equivalent statements is crucial.

Solutions, cont'd.

Problem C5, cont'd.

c. This activity requires level 1-type of thinking (analyzing and reasoning about different properties for a whole class of shapes; i.e., rectangles). Moving into level 2 would involve directing students toward thinking about the equivalence of the listed properties and focusing on the information necessary to define a rectangle. The definition of a rectangle as a quadrilateral with all four angles equal to 90º would be an example. This problem is very similar to the definition activities we did in Session 3, Part C.

d. To extend students' thinking, you could do a similar activity with a more complicated figure. (Defining a circle, for example, is a more challenging task, as is defining any sort of three-dimensional figure.) You can also ask students to focus on the minimal information necessary. (For example, "a quadrilateral with three right angles" is enough—the fourth angle is forced to be 90º as well. Similarly, "a parallelogram with one right angle" is enough.)

e. One possible lesson: As a whole class, brainstorm several properties of rectangles. Generate a long list on the board, and really push students to offer obvious (four sides) and not as obvious (congruent diagonals) properties. You can ask leading questions ("What about the diagonals?") and give them time to think and sketch as you generate the list. When the list is fairly long, go item by item and put stars next to things that students think are true only of rectangles and nothing else. (Don't worry if they are not correct; refining their ideas is part of the activity.) Explain that a good definition will describe every rectangle in the world and nothing else. Ask students to spend several minutes drawing figures that fit the properties with stars, trying specifically to draw things that are not rectangles. End this part of the activity by refining which statements have stars by them based on what students have found. You can end the activity by assigning groups of students to come up with alternate definitions based on combining the remaining statements. Their job is to combine two or three of the non-starred statements into something that describes only rectangles. You can end the activity by writing several of the definitions on the board. You might want to talk about why we want alternate definitions (sometimes it's convenient to think about shapes in different ways, and it's useful to have definitions that relate to angles, definitions that relate to sides, definitions that combine these, to make our work easier). You can also provide some standard definitions from textbooks and dictionaries, for students to see how their definitions compare, and to see that even the "real" definitions will vary from source to source.

Problem C6.

a. On the face of it, this seems like a straightforward task of relating inputs (number of sides of a polygon) and outputs (the total number of sides of the resulting two figures) of a function (drawing a segment). Once you investigate it, though, you find that it is more complicated than it appears on the surface. This is a problem in examining extreme cases (minimum and maximum number of sides possible as a result), and in considering what causes the extreme cases (connecting vertex to vertex, side to side), and so on. Further examination reveals that there are only three possible cases (connecting vertex to vertex, side to vertex, and side to side), and the results begin to emerge into a pattern.

b. There is no prerequisite knowledge necessary other than the ability for students to identify and count the number of segments on a polygon. To be successful, though, students should have some experience with open-ended problems, creating and testing cases, creating tables to track data, and working with non-obvious number patterns.

   This task prepares students for thinking through a complicated situation, for relating different areas of mathematics (functions, number patterns, and geometry), and for explaining their results in informal arguments.

c. In this case, students will be able to explore the problem and come up with their own generalizations. As a result, they will be able to construct an informal argument that supports their explorations. On the surface, this appears to be a level 1 task, with an easy entry for students who might be struggling to move towards deduction. It does, however, bridge to level 2 thinking.

Solutions, cont'd.

Problem C6, cont'd.

d. To extend student thinking, you could lead into activities like the cross sections of three-dimensional figures (see Session 9, Part C). Questions here could include both what cross-sectional shapes are possible and the total number of faces of the resulting two figures. (This is a much more difficult task than the one presented in two dimensions!) Another extension is thinking about how segments can split regions apart more generally. For example, start with a circle: One line can split it into two regions. Two lines can split it into either three or four regions (how?). What are the possibilities for three, four, or five lines? This is an interesting problem, because if you focus on the maximum number of regions, there is a rather simple pattern at first, but it breaks down when you look at the case of six lines!

e. One possible lesson: Draw several triangles on the board. Select students to come up and draw a segment across each triangle. When they have finished, as a class, count up the total number of sides of the resulting two figures. (It needs to be clear to everyone that the drawn-in segment is counted twice—once for each of the new figures.) If students have not done so themselves, make sure that some of the segments go side to side and some go side to vertex. You can draw another triangle, say you have another way to create a segment, and demonstrate it. When the task is clear—finding all the possible ways to draw a segment and then counting up the total number of sides—pass out sheets to each group of students. The sheets should have several copies of each starting shape for students to work with and a place for them to keep track of their results. To wrap up the lesson, write the names of shapes on the board (triangle, quadrilateral, pentagon, hexagon) and, based on the students' data, fill in the totals that were possible for each one. Allow for explanation and examples, for example, if one group found something that another group did not. Try to draw out of students a description of the three possible segments and how they affect the total number of sides.

Notes