Session 2

Triangles and Quadrilaterals

Key Terms for This Session

Previously Introduced

- altitude
- perpendicular bisector

New in This Session

- acute triangle
- congruent
- congruent triangles
- equilateral triangle
- isosceles triangle
- obtuse triangle
- right triangle
- scalene triangle
- side-angle-side (SAS) congruence
- side-side-side (SSS) congruence
- similar triangles
- triangle inequality

Introduction

In this session, you will build triangles and quadrilaterals to explore their properties. Classification is an important part of geometry and other areas of mathematics. By creating and manipulating triangles and quadrilaterals, you will develop a sense of logical classifications and how different figures are related.

For information on required and/or optional materials for this session, **see Note 1.**

Learning Objectives

In this session, you will learn the following:

- How to classify triangles according to some of their features
- A rule that describes when three given lengths will make a triangle and when they will not
- A rule that describes when four given lengths will make a quadrilateral and when they will not
- How to use what you've learned about geometry and the properties of figures to help with a construction task

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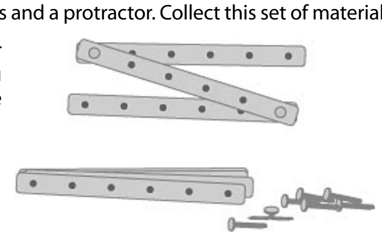
Note 1.

Materials Needed:

- linkage strips
- scissors
- box of toothpicks
- bag of mini marshmallows
- meter stick or ruler

For homework, you'll also need a ruler marked in inches and a protractor. Collect this set of materials for each group or individual working alone.

You can make your own linkage strips from stiff paper with evenly spaced holes and paper fasteners, or you can purchase them from ETA/Cuisenaire under the name Polystrips.



Polystrips
ETA/Cuisenaire
500 Greenview Court
Vernon Hills, IL 60061
800-445-5985/847-816-5050
800-875-9643/847-816-5066 (fax)
(Product not available online)

Part A: Different Triangles (20 minutes)

Drawing Triangles

In the following problems, you will draw a variety of triangles and take note of their differences. **[See Note 2]**

Problem A1.

a. Draw any triangle on your paper.

b. Draw a second triangle that is different in some way from your first one. Describe in just a word or two how it is different.

Problem A2. Draw a third triangle that is different from both of your other two. Describe how it is different.

Problem A3. Draw two more triangles, different from all the ones that came before.

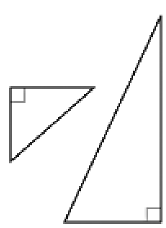
Problem A4. To make "different" triangles, you have to change some feature of the triangle. Make a list of the features of triangles that you changed.

Classifying Triangles

The following shows how triangles can be classified according to some of their features:

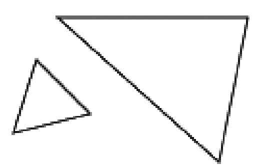
Angles:

Right Triangles



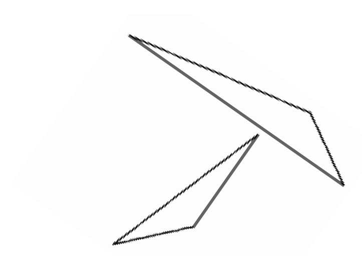
One right (90°) angle

Acute Triangles



All angles less than 90°

Obtuse Triangles



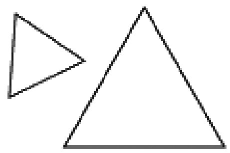
One angle more than 90°

Note 2. If you're working in a group, first work on Problems A1-A5 individually. Afterwards, break into small groups to compare the lists of the features of triangles that everyone created.

Part A, cont'd.

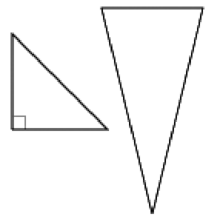
Sides:

Equilateral Triangles



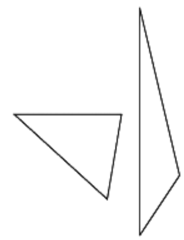
All three sides are the same length.

Isosceles Triangles



Two sides are the same length.

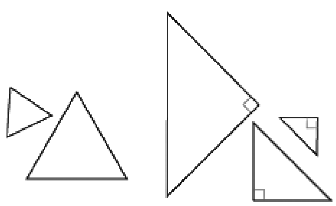
Scalene Triangles



All three sides are different lengths.

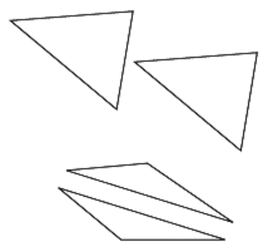
Size:

Similar Triangles



Same shape, possibly different size

Congruent Triangles



Same size and shape

Problem A5. More than one feature can be combined into a triangle. Decide which of the following combinations are possible. If the combination is possible, draw a sketch on a piece of paper. If not, explain why not.

a. a scalene right triangle

b. an isosceles right triangle

c. an equilateral right triangle

d. a right triangle that is also obtuse

Part B: Linkage-Strip Constructions (40 minutes)

Constructing Triangles

A triangle has three sides, but not just any set of three lengths will make a triangle. Use the linkage-strip activity to answer Problems B1-B5. **[See Note 3]**

Begin with three linkage strips of any length. Try to connect the linkage strips at their endpoints to form a triangle. Do all three sides connect? Try this with linkage strips of different lengths to test when a set of three lengths will form a triangle. When you've built a triangle, see if you can deform it (change its shape.)

Try It Online! **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Geometry* Web site at www.learner.org/learningmath and find Session 2, Part B.

Problem B1. Fill in the table below. Try to build triangles with the given lengths. Write "yes" or "no" in the fourth column of the table to indicate whether you can or cannot make a triangle from those three lengths. Experiment with different sets of lengths. When you build a triangle, see if you can deform it (change its shape) into a different triangle while keeping the side lengths the same. **[See Note 4]**

Side A	Side B	Side C	Is it a triangle?	Can it be deformed?
4 units	4 units	4 units		
4 units	3 units	2 units		
3 units	2 units	1 unit		

Problem B2. Suppose you were asked to make a triangle with sides 4, 4, and 10 units long. Do you think you could do it? Explain your answer. Keep in mind the goal is not to try to build the triangle, but to predict the outcome.

Problem B3. Come up with a rule that describes when three lengths will make a triangle and when they will not. Write down the rule in your own words.

Note 3. For this activity, you may make your own strips to answer Problems B1-B5.

Note 4. If you are working in groups to construct triangles and then, in Problem B6, quadrilaterals, discuss the rules you came up with for instances in which three lengths make a triangle and in which four lengths make a quadrilateral.

Sometimes people may be confused about the question of whether or not you can "deform" one triangle into anther one. It helps to build another figure—use five sides so as not to give away the answers for the quadrilaterals in Problem B6—and see how easy it is to push on the sides and angles to form many different pentagons with the same five sides. Then, although the materials may allow a little bit of "wiggle," it becomes clear whether or not you can deform a triangle into a completely different one.

Part B, cont'd.

Problem B4. Suppose you were asked to make a triangle with sides 13.2, 22.333, and 16.5 units long. Do you think you could do it? Explain your answer.

Video Segment (approximate time: 9:25-12:11): You can find this segment on the session video approximately 9 minutes and 25 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Vicky and Lolita write a rule that describes when three lengths will make a triangle. Watch this segment after you have completed Problems B1-B4, and compare your rule with that of the onscreen participants.

What was the first rule that Vicky wrote? How did she and Lolita revise this rule? How does this rule compare with the one that you wrote?

Video Segment (approximate time: 12:50-13:32): You can find this segment on the session video approximately 12 minutes and 50 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, Kent describes a different rule for when three lengths will make a triangle. Watch this segment after you have completed Problems B1-B4, and compare your rule with Kent's.

What was Kent's rule? How is it different from Vicky and Lolita's rule? How does this rule compare with the one that you wrote?

Problem B5. Can a set of three lengths make two different triangles? **[See Tip B5, page 43]**

Part B, cont'd.

Constructing Quadrilaterals

Use the linkage-strip activity to answer Problems B6-B9. This time you will try to connect four linkage strips of different lengths to build quadrilaterals.

Try It Online! **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Geometry* Web site at www.learner.org/learningmath and find Session 2, Part B.

Problem B6. Fill in the table below. Use the linkage-strip activity to try to build quadrilaterals with the given lengths. Write "yes" or "no" in the fifth column of the table to indicate whether or not you can make a quadrilateral from those four lengths. Experiment with different sets of lengths. When you build a quadrilateral, see if you can deform it into a different quadrilateral with the same side lengths.

Side A	Side B	Side C	Side D	Is it a quadrilateral?	Can it be deformed?
4 units	4 units	4 units	4 units		
4 units	3 units	2 units	2 units		
3 units	2 units	1 unit	1 unit		
4 units	1 unit	2 units	1 unit		

Problem B7. For some of the lengths above, can you connect them in a different order to make a different quadrilateral? If so, which ones? How is this different from building triangles?

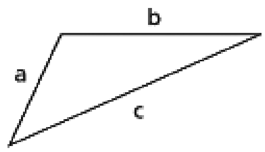
Problem B8. Come up with a rule that describes when four lengths will make a quadrilateral and when they will not. Write down the rule in your own words. (You may want to try some more cases to test your rule.)

Problem B9. Can a set of four sides of the same length make two different quadrilaterals?

Part B, cont'd.

Properties of Triangles

The triangle inequality is a famous result in mathematics. It says that for three lengths to make a triangle, the sum of any two sides must be greater than the third side. Often you will see a picture like this, where a, b, and c represent the three lengths of the sides.



$a + b > c$

The triangle inequality is the mathematical statement of the old adage, "The shortest distance between two points is a straight line." If you don't travel along the straight line, you travel two sides of a triangle, and that trip takes longer.

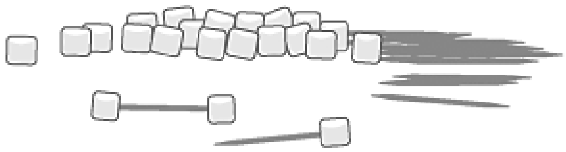
You have also probably found that triangles are rigid. That is, if a set of lengths makes a triangle, only one triangle is possible. You can't push on the vertices to make a different triangle with the same three sides. Triangles are the only rigid polygon, which makes them quite useful for construction.

This property is often abbreviated as SSS (side-side-side) congruence. If the three sides of one triangle have the same lengths as the three sides of another triangle, then the two triangles are congruent. That is, they have exactly the same size and shape. All of the angle measurements will match, as will other measurements, such as their areas, the lengths of the corresponding altitudes, and so on. If you cut the two triangles out from a piece of paper, you could fit one exactly on top of the other.

Part C: Building Towers (45 minutes)

In Part C, you will apply what you've learned about the properties of figures to a construction task.

Problem C1. Gather toothpicks and mini marshmallows, or other connectors. Your job is to work for 10 minutes to build the largest freestanding structure you can. "Freestanding" means the structure cannot lean against anything else to keep it up. At the end of 10 minutes, stop building, and measure your structure. **[See Note 5]**



Problem C2. What kinds of shapes did you use in your structure? Which shapes made the building stronger? Which shapes made the building weaker?

Problem C3. If you had the chance to build the structure again, what would you do differently?

Video Segment (approximate time: 18:50-21:42): You can find this segment on the session video approximately 18 minutes and 50 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

In this video segment, participants attempt to build freestanding structures with toothpicks and mini marshmallows. Watch this segment after you have built your own freestanding structure, and compare your strategy with that of the onscreen participants.

What kinds of shapes did the onscreen participants use in their structures the first time? How did some of the groups revise their strategy? How did your structure compare with those of the onscreen participants?

Problem C4. Get another set of building materials and take an additional 10 minutes to create a new freestanding structure. Your goal this time is to build a structure *taller* than the one you made before.

Note 5. If you are working in groups to build your structures, at the end of 10 minutes, measure your structure, and then compare the structures built by the different groups. Consider the following questions:

- For the tallest structures in the room, what kinds of strategies did the creators use?
- What kinds of structures had trouble standing up? Which ones were stronger?
- What kinds of shapes were stronger in keeping the buildings up? Can you explain why?

Homework

Problem H1. More than one feature can be combined into a triangle. Decide which of the following combinations are possible. If the combination is possible, draw a sketch. If not, explain why not.

a. a scalene acute triangle
b. an isosceles acute triangle
c. an equilateral acute triangle
d. a scalene obtuse triangle
e. an isosceles obtuse triangle
f. an equilateral obtuse triangle
g. an equilateral triangle that is also isosceles

Problem H2. A certain quadrilateral has one diagonal that is 2 inches long and another diagonal that is 3 inches long. A diagonal is a line segment connecting any two non-adjacent vertices.

a. Draw two such quadrilaterals. What, if anything, do they have in common? How are they different?
b. Draw such a quadrilateral where the diagonals bisect each other but are not perpendicular. What does it look like?
c. Draw such a quadrilateral where the diagonals are perpendicular but do not bisect each other. What does it look like?
d. Draw such a quadrilateral where the diagonals bisect each other and are perpendicular. What does it look like?

Problem H3. Create a quadrilateral with diagonals that are the same length and bisect each other. What kind of quadrilateral is it? Can you explain why?

Problem H4. Create a quadrilateral with diagonals that are the same length, bisect each other, and are perpendicular. What kind of quadrilateral is it? Can you explain why?

Homework, cont'd.

Problem H5. For each part below, draw two different triangles that fit the information given. What do you notice?

a. One side is 2 inches long; another side is 3 inches long. The angle between them is 45º.

b. One side is 2 inches long; another side is 3 inches long. The angle between them is 75º.

c. One side is 2 inches long; another side is 3 inches long. The angle between them is 90º.

[See Tip H5, page 43]

Problem H6. You have already seen the SSS (side-side-side) congruence test for triangles: If the three sides of one triangle have the same lengths as the three sides of another triangle, then the two triangles are congruent. That is, they have exactly the same size and shape. Describe and name a new congruence test based on your work in Problem H5.

Problem H7. For each part below, draw two different triangles that fit the information given. What do you notice?

a. The three angle measures are 45°, 45°, and 90°.

b. The three angle measures are 60°, 60°, and 60°.

c. The three angle measures are 100°, 30°, and 50°.

[See Tip H7, page 43]

Problem H8. Is there an angle-angle-angle (AAA) congruence test for triangles? That is, if the three angles of one triangle have the same measures as the three angles of another triangle, are the two triangles necessarily congruent? Explain your answer.

[See Tip H8, page 43]

Suggested Reading

This reading is available as a downloadable PDF file on the *Geometry* Web site.
Go to www.learner.org/learningmath.

Steen, Lynn Arthur (1990). Pattern. In *On the Shoulders of Giants: New Approaches to Numeracy*. Edited by Lynn Arthur Steen (pp. 1-10). Washington, D.C.: National Academy Press.

Tips

Part B: Linkage-Strip Constructions

Tip B5. To answer this question, you will need to know what it means for two triangles to be "different." One definition says that triangles that are "different" cannot have the exact same size and shape. Rotating or reflecting a triangle with the same size and shape does not produce a "different" triangle.

Homework

Tip H5. Are the triangles different or congruent? If you think they are congruent, try to draw a triangle that fits the description but is not congruent.

Tip H7. Are the triangles different or congruent? If you think they are congruent, try to draw a triangle that fits the description but is not congruent.

Tip H8. Use your work in Problem H7 to answer this question.

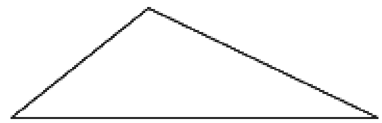
Solutions

Part A: Different Triangles

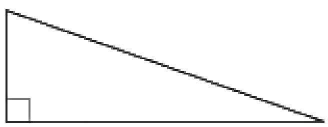
Problem A1.

a. Answers will vary. The following is one example:

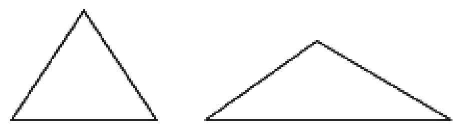
b. Answers will vary. One example is a triangle whose side lengths are longer.



Problem A2. Answers will vary. One example is a triangle that, unlike the previous two, has one right angle.



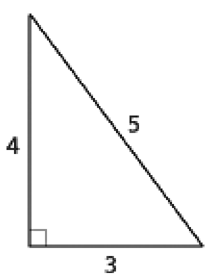
Problem A3. Answers will vary. Some examples are the following:



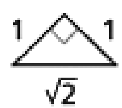
Problem A4. Answers will vary, but generally, either the measures of angles or lengths of sides need to be changed to make "different" triangles.

Problem A5.

a. This is possible.



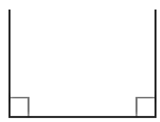
b. This is possible.



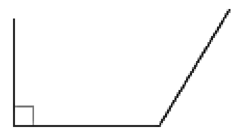
Solutions, cont'd.

Problem A5, cont'd.

c. This is impossible. Inside a triangle, equal angles correspond to equal sides, so for a right triangle to be equilateral, it would have to have three right angles. Two right angles next to each other, however, form parallel lines, which would mean it would not be possible to complete such a triangle.



d. This is impossible. The picture below shows the right angle and obtuse angle next to each other, with the side in between laid out horizontally. The side that extends from the right angle is vertical, while the side that extends from the obtuse angle is pointed *away from* the side that extends from the right angle. Because these sides must be connected to form a triangle, this kind of triangle is impossible to make.



Part B: Linkage-Strip Constructions

Problem B1. Here is the table filled in for the triangles with given lengths and other sample triangles.

Side A	Side B	Side C	Is it a triangle?	Can it be deformed?
4	4	4	Yes	No
4	3	2	Yes	No
3	2	1	No	N/A
4	3	2	Yes	No
1	2	4	No	N/A
2	4	4	Yes	No
3	1	1	No	N/A
2	3	3	Yes	No
2	4	2	No	N/A

Other answers will vary individually, but no triangle will be deformable.

Problem B2. No, this is not possible. If we attach the two sides of lengths of 4 units to the endpoints of the side of length 10, the first two sides will not meet at a point to create a triangle. Together they are too short.

Problem B3. Three lengths can form a triangle only if the sum of the lengths of any two sides is greater than the length of the third side.

Problem B4. Yes. Because the sum of lengths of any two sides is greater than the length of the remaining side, the two sides will be able to meet at a point and create a triangle when attached to the endpoints of the third side.

Solutions, cont'd.

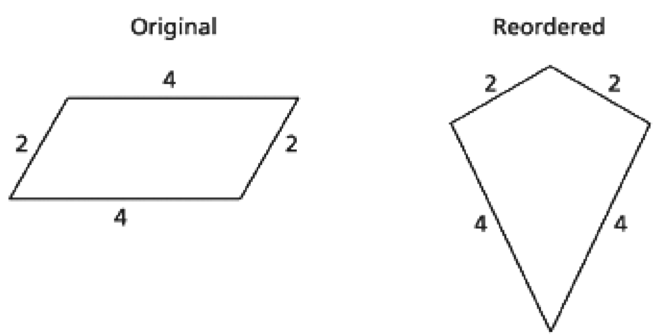
Problem B5. No, three fixed lengths determine one and only one triangle. This is demonstrated by the fact that none of the triangles found in Problem B1 can be "deformed" into a different shape.

Problem B6. Here is the table filled in for the triangles with given lengths and other sample triangles.

Side A	Side B	Side C	Side D	Is it a quadrilateral?	Can it be deformed?
4	4	4	4	Yes	Yes
4	3	2	2	Yes	Yes
3	2	1	1	Yes	Yes
4	1	2	1	No	N/A
1	1	1	4	No	N/A
2	2	2	2	Yes	Yes
1	4	3	1	Yes	Yes
1	3	3	4	Yes	Yes
2	3	4	1	Yes	Yes
4	1	1	2	No	N/A

Other answers will vary individually, but all quadrilaterals will be deformable.

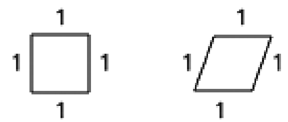
Problem B7. As long as no more than two sides of a quadrilateral are equal in length, we can reorder the way the sides are connected and obtain a different quadrilateral. This is not the case with triangles: If we reorder the sides, we get the same triangle.



Problem B8. Four lengths can form a quadrilateral as long as the sum of the lengths of any three sides is greater than the length of the fourth side.

Solutions, cont'd.

Problem B9. Yes. For example:



Also, if the sides are not the same length, ordering them differently will produce different quadrilaterals.

Part C: Building Towers

Problem C1. Constructions will vary.

Problem C2. Answers will vary depending on the construction, but in general, structures whose surfaces are defined by quadrilaterals will be less sturdy than those whose surfaces are defined by triangles.

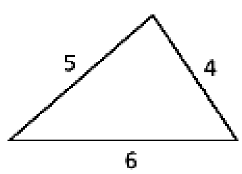
Problem C3. Answers may vary, but as we've learned, triangles are more rigid than quadrilaterals. Similarly, using a wide base and building "up" with triangles is the best approach. Builders find that if you want a square- or rectangular-shaped building, you must build triangular supports into the "walls."

Problem C4. Constructions will vary.

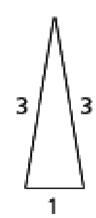
Homework

Problem H1.

a. It is possible. For example:

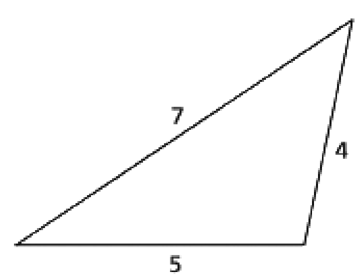


b. It is possible. For example:



c. It is possible. For example:

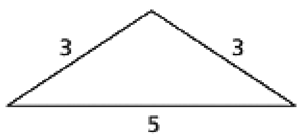
d. It is possible. For example:



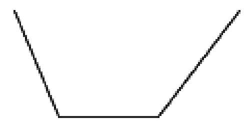
Solutions, cont'd.

Problem H1, cont'd.

e. It is possible. For example:



f. This is impossible, because equal sides correspond to equal angles. This would mean that a triangle would have two consecutive obtuse angles. The sides extending from these two angles could not be connected for the same reason as in Problem A5, part (d).

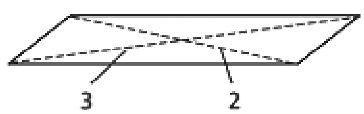


g. It is possible. In fact, all equilateral triangles are also isosceles triangles.

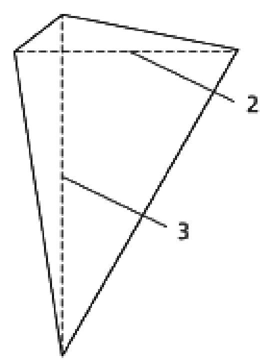
Problem H2.

a. Constructions and answers will vary. But if the diagonals have different lengths (such as 2 and 3 units), then the figure can never be a square or a rectangle.

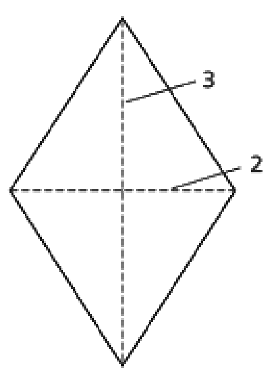
b. Constructions will vary, but the figure will always be a parallelogram. One possible example is the following:



c. Constructions will vary, but the figure will either be a kite or a random quadrilateral. One possible example follows:

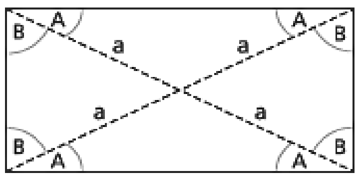


d. In this case, there is only one possibility—a rhombus:

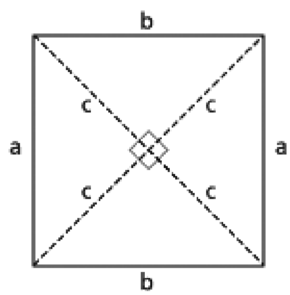


Solutions, cont'd.

Problem H3. You should find that the result must be a rectangle. Here's one explanation for that fact: The sum of the interior angles in any quadrilateral is 360°. Because the line segments marked "a" are all equal, the angles opposite to them inside the respective triangles are equal. Therefore, the sum of angles $4A + 4B = 360°$; i.e., $4(A + B) = 360°$; i.e., $A + B = 90°$. Hence, all of the interior angles are right angles, and the quadrilateral is indeed a rectangle.



Problem H4. The quadrilateral in question is a rectangle as described in the solution to Problem H3. In addition, two adjacent isosceles right triangles with hypotenuses a and b respectively are congruent since they have congruent legs, and the congruent (right) angles between them. So we must have a = b, and therefore the quadrilateral is a square.



Problem H5. In parts (a)-(c), it is impossible to draw two different triangles. In other words, if we fix two sides and the angle between them, we uniquely determine a triangle.

Problem H6. This type of congruence can be called SAS (side-angle-side) congruence: If two triangles have two sides equal in length, and the angles between those sides are equal in their degree measure, then the two triangles are congruent.

Problem H7. In parts (a)-(c), we can create two or more distinct triangles by keeping the angles fixed and changing the lengths of the sides. For example, we can build a second triangle where each side is twice as long as the original and the angles will remain the same size.

Problem H8. No. Problem H7 shows that two triangles can have the same size angles without being congruent. The triangles appear to have the same overall shape, but they might be larger or smaller than the original. They are not congruent, but they do have a relationship. They are called similar.

Notes