Bishop's lessons had just turned their ideas about triangles, figuratively and literally, upside-down. A triangle could be tilted and still be a triangle. A triangle could be "squished" and still be a triangle. A square could be tilted and still be a square. And so, by analogy, why could not a square, too, be squished and still be a square? These children were not yet aware that unlike the angles of a triangle, the angles of a square cannot, by definition, vary. Bishop knew what her next lessons must address.

As this episode suggests, the shift from a gestalt-centered classification of shapes to the study of properties is a complex process. Even as students begin to realize that they need to look at the number of sides, the lengths of the sides, and the angles of a polygon, they must attend to subtleties and nuances. Furthermore, as Doris Flynn describes in the following episode, movement from one "level" to the next does not come in one sharp and final break; instead, for an extended period, children may oscillate between older and newer conceptions.

Like Sanford and Bishop, Flynn records the day on which she realized that her third-grade students had a very particular image of what a triangle could be. Given an assortment of triangles drawn on the chalkboard (see **fig. 4**), all twenty-five children agreed, "Only one of them [shape C] is a triangle." Flynn asked the class, "What about these other shapes?" She writes about how they responded.

"No," Susan said. "You might think that some of the others are triangles, too. But if you think any of those are **real** triangles, then you are wrong." The class was very eager to say all the reasons why they were not **real** triangles. They were too pointy, too long, too tipped, too different, or, worst of all, judging by their discomfort with this characteristic, "going the wrong way."

The children took quite a while trying to say what it was about a shape that makes it a triangle. Finally, we got to "three sides and three corners that look **right**." At that point, we went to the math textbook in the closet and checked the glossary for a definition. The Addison-Wesley textbook said, "A plane figure with three segments as sides." I was pleased to see in the book a drawing that looked like one of the triangles on my board. [See **fig. 4**, shape D.]

Now the kids were buzzing. If all you need are three sides, the class was ready to accept the group of triangles that had the bottom in the "right place" also. Gradually, with more discussion, the class tested out each shape and came to an agreement that all of them fit the rules, so they must all be triangles. There was an atmosphere of amazement. Some looked quite proud of themselves for finding out something new. Kathy even said something like, "Who would have ever thought that things like that could be triangles?"

We might think that such an extensive discussion about triangles ought to have firmly planted new notions in Flynn's students' heads. After all, the children themselves acknowledged that they had just learned something important and could talk about how surprising it was. However, several weeks later, Flynn realized that the issue was not yet settled.

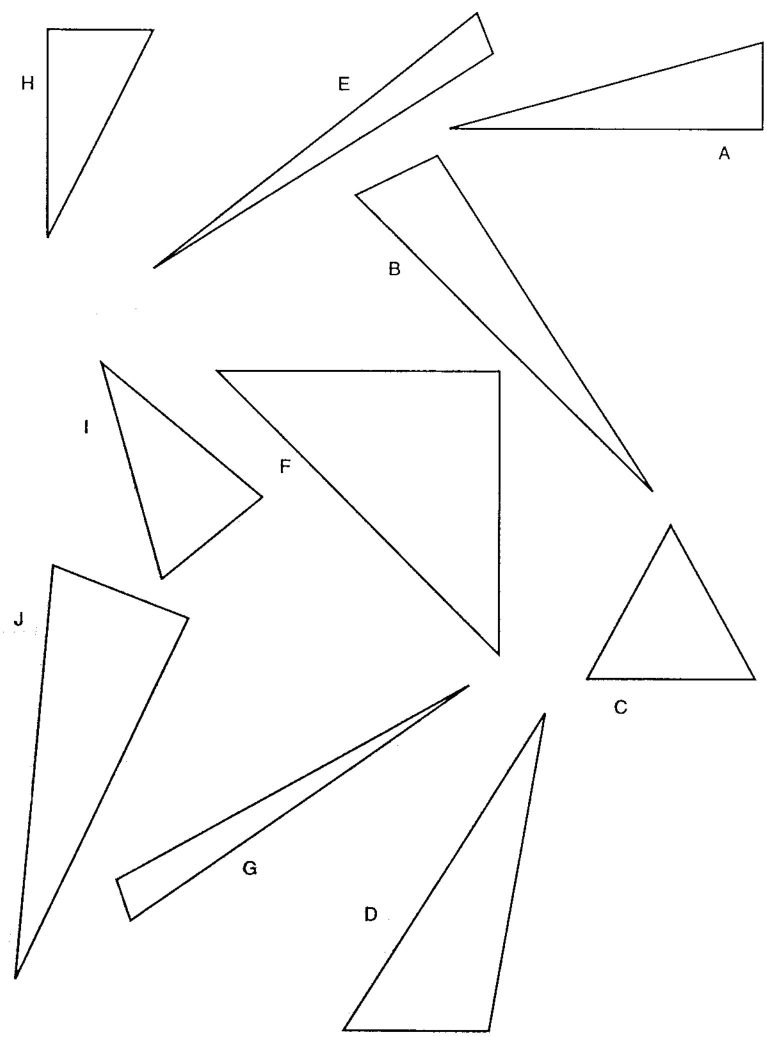
I was having a conversation with one of my rather competent math students when he said of a triangle that it could get to be a triangle if the sides were moved a little. I immediately thought, hold everything, we've covered that! Now I wondered what the class actually gained from consulting the book and finding the true definition.

A few weeks later, I put eight assorted triangles on the board and asked everyone to secretly write to me their current thinking about the shapes; I never said the word *triangle*. The results showed that many students had not moved from their earlier thinking. The standard well-known, base-on-the-bottom, equal-sided triangle was a triangle. Those pointing any direction but up could perhaps be "fixed" to be triangles if they weren't too long, too pointy, or "too weird." Too weird was explained to me as too out of balance, or too much difference in the size of the sides.

As engaging and student centered as Flynn's previous lesson had been, the understandings that the children seemed to exhibit by the end of class that

FIGURE 4

Third graders claimed that only one of these shapes, shape C, was a triangle.



day had not remained fixed. Was it that they had forgotten? Were they confused? Or had she misassessed the extent of their understanding of that lesson?

Given Flynn's discovery, we might expect her to have felt dismay. Were her students incapable of learning? Had her teaching been ineffective? Or was something else going on? In fact, Flynn was not dismayed; she was curious. "What was difficult about these ideas for her students?" she wondered. And rather than ponder this question alone, she put it directly to the source:

> I told the class that I had been spending years trying to figure out how children learn ideas in mathematics. I was always trying to get to the bottom of ideas that are hard for kids and why those ideas are hard.
>
> I was surprised that the conversation seemed to move to a higher level and that the children were eager to think about my question with me. Lynn was able to say that it was really hard to think of the different triangles as triangles because she felt like she really knew one way best and she had known it all her life. Wow! Cathy said that she felt like she really had to break out of the thinking she did before and let some new ideas into her head. Wow again!
>
> Others talked readily about really taking a fresh look at things. Some said old stuff needs to move over to fit new stuff you found out. And then some pointed out that the "regular" triangle might always feel like "more of a triangle" than some of the other triangles. I think they are right.

These young children articulately described their experience of disequilibrium. Hard work was needed to "break out of" old ways of thinking to "fit in new stuff you find out." Their words help us understand why a few well-designed, student-centered lessons cannot guarantee that students will move smoothly and finally from a gestalt approach to classification of geometric shapes to classification by properties.

Episode Writing: A Mechanism to Pool Teachers' Learnings

Although these episodes illustrate some of the challenges that teachers face when dealing with geometry in the elementary grades, they should not be discouraging. The children described here are good reasoners, actively engaged with the ideas on the table. They help their teachers identify the mathematical issues needing attention—where the work needs to be done. Thus, these stories help in designing mathematics instruction that elicits students' ideas and builds on those ideas to develop stronger, more robust conceptions.

These episodes also demonstrate the importance of teachers' finding mechanisms to pool what they learn from their own teaching. By filling out the picture of what is involved in helping students advance their geometric understandings, teachers can become better prepared to support that process. With an appreciation for the complexity of the content, teachers might also have more nuanced images of what constitutes progress.

Episode writing is not meant to replace more conventional forms of research. Rigorous and systematic data collection and analysis are necessary. However, as a complement to such work, teachers' episode writing—with a particular focus on students' mathematical conceptions—can contribute to the knowledge base of the field and help bridge the gulf between research and practice.

Participants in the TBI project have found episode writing to be particularly suited to classroom teachers. Acknowledging that it does take time—a scarce commodity in many teachers' lives—some teachers have likened it to sticking to an exercise regimen. Once they have made the commitment, the exercise is satisfying and worthwhile. Teachers report that they appreciate the excuse to clear their minds of their many concerns while concentrating for an hour on just one small part of the day—What did each child say? What did he mean by that? What is the idea she is chewing on? And they find that episode writing attunes the ear. They become better listeners, and, as a consequence, their students become more thoughtful mathematical discussants.

The episodes presented in this article contribute to the emerging picture of children's developing understanding of geometry and the kinds of teaching that can support it. Other sets of episodes produced by the TBI project illustrate students' developing understandings in such areas as place value and computation, the meanings of operations with whole numbers and fractions, measurement, data representation, and early algebraic thinking (Bastable and Schifter, in press; Schifter, in press; Schifter, Bastable, and Russell 1998; Schifter and O'Brien 1997; Sweeney 1998). They offer a beginning. Much remains to be done.

Appendix

A Sample Episode from Grade 4

The children are looking at a paper with nine quadrilaterals. [See fig. 5.] I ask them to think about the shapes, talk with their neighbors, and decide which shapes they would consider squares and which they thought were rectangles.

In planning for my work with this group of eight fourth graders, I had decided to start with this activity, which would force the students to create definitions for both rectangles and squares. I hoped to challenge their present, perhaps simplistic, definitions; those that they had carried with them for several years. I predicted, and strongly hoped, that looking closely at quadrilaterals would bring up other, richer issues, such as area, perimeter, angles, and relationships among various quadrilaterals. Immediately these issues and others begin to be evident.

Josh: I see one that is a rectangle and square put together. [He points to figure F, a rectangle whose pairs of sides are close in length, making it look squarish.]

Adam: What is D? It's not a square or a rectangle.

Brian: Would B be a square 'cause it's not straight? [B is, in fact, a square drawn slightly at a slant on the paper.]

Teacher: What do you think?

Brian: Yes! [He tilts his paper as if to straighten the square and confirms himself confidently.]

After only five minutes of work, someone announces, "I'm done!" And, predictably, several other voices join in. I suggest that they take a few minutes to check their ideas and to talk with the others about the shapes. Thankfully, my brief refocusing works, and I hear more discussion about the quadrilaterals.

While the work continues, Juliana loudly states, "D isn't either a square or a rectangle." I find it interesting that she is repeating what Adam said earlier, yet it sounds as if she just came to that conclusion. She must not have been ready to think about Adam's comment earlier on in the class. Josh, it seems, however, has been thinking about D since Adam brought it up. He said, "I think it's a rectangle 'cause all the sides are different." I am able to squeeze in a question even though suddenly the conversation is moving quickly: "What do you mean, Josh, that all the sides are different?" "This side is different from this side is different from this side is different from this side," he explains as he points at each side of the trapezoid.

Brian: I measured D with my pencil, and these three sides are the same length [he points to the top and the two sides], but this one [the bottom] is much longer. [He holds his thumb and forefinger apart about an inch.]

Emily: The bottom is a lot bigger. About an inch.

The rest of the group is stimulated by this conversation and request rulers. Some discussion is heard about whether to round off, which lines are fractions and which are whole inches, and how exact the measurements should be. I had expected that most of these issues would come up, but I had decided beforehand that I wouldn't pursue them at this time, so I let them slide by.

Teacher: [After some time for measuring and talking] What did you find out about D?

Adam: The left side is longer and steeper, and the right side is shorter and straighter.

Brian: The top is 1 1/2 inches, and the bottom is 2 1/2 inches.

Juliana: It's not a square or a rectangle.

Teacher: Juliana, that's the same statement you made earlier. What makes you so sure about it? [She shakes her head as if she's not able to explain, not as if she's unsure.]

Chris: The left side is 1 3/4 inches, and the right side is 1 5/8 inches. [He gets up and shows each person where 5/8 is on the ruler.]

We check Chris's information with Adam's description to see if they support each other. They do, and the students are satisfied by this, nodding and smiling.

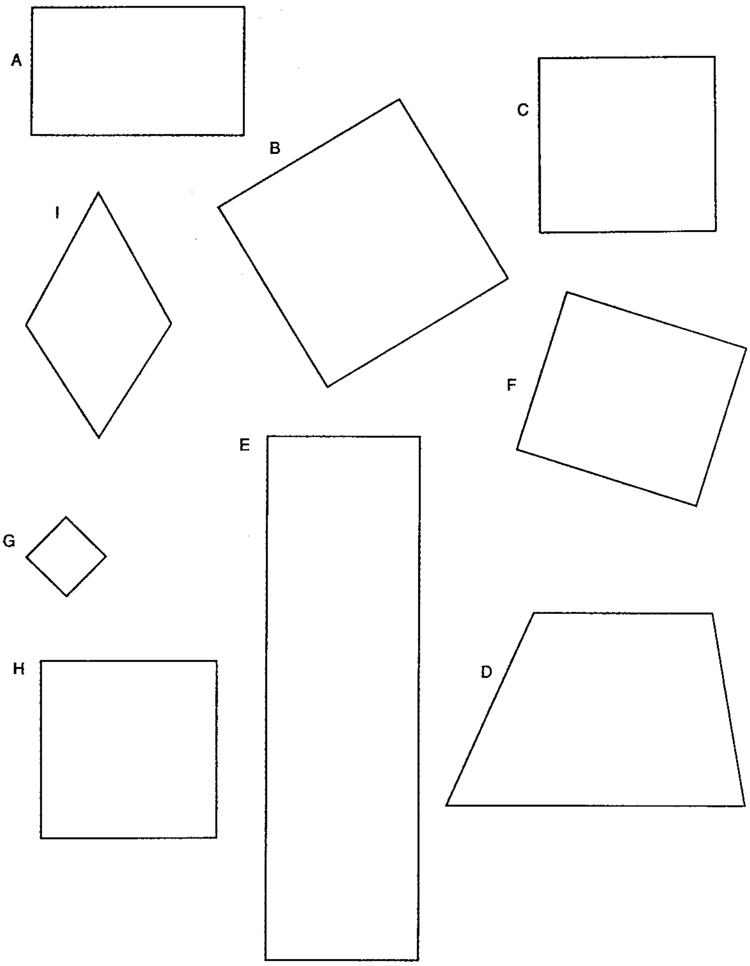
Although the students have observed much about the trapezoid, the discussion needs to be redirected now. I'd like to return to my original plan for the day, so I try encouraging the comparison of shape D with a rectangle. I ask, "Who has a definition for a rectangle?"

Emily: The left and right sides are longer and the same. The top and bottom sides are shorter and the same.

Josh: [Repeats Emily's definition and compares it with D]: D has no sides that are the same, so it can't be a rectangle.

FIGURE 5

Fourth graders discussed this page of quadrilaterals.



All the children either verbally or physically show agreement with Josh. A couple of them restate Josh's comment in their own words, and then it feels as if it's time to move on. I hold up the paper with the quadrilaterals and say, "Emily's definition of a rectangle makes me think of shape E. Is that a rectangle?" E is a narrow, vertical rectangle. The students agree. "So what is E if I hold it this way?" I turn the paper so E is horizontal. Several of the children stare at me with their jaws dropped, as if stunned and very confused. Three of them shake their heads and say, "No, it can't be a rectangle then." I, too, am surprised. I had expected fourth graders to be clear about the constancy of a shape despite its orientation.

I have begun to feel more confident about leaving students in a point of confusion for a time, so I decide to stop here for today. I invite the children to talk with others about rectangles and to think about what they'd call E when it's turned onto its side.

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