come as no surprise that the essays themselves are replete with interconnections, both in deep structure and even in particular illustrations. Some examples:

MEASUREMENT is an idea treated repeatedly in these essays. Experience with geometric quantities (length, area, volume), with arithmetic quantities (size, order, labels), with random variation (spinners, coin tosses, SAT scores), and with dynamic variables (discrete, continuous, chaotic) all pose special challenges to answer a very child-like question: "How big is it?" One sees from many examples that this question is fundamental: it is at once simple yet subtle, elementary yet difficult. Students who grow up recognizing the complexity of measurement may be less likely to accept unquestioningly many of the common misuses of numbers and statistics. Learning how to measure is the beginning of numeracy.

SYMMETRY is another deep idea of mathematics that turns up over and over again, both in these essays and in all parts of mathematics. Sometimes it is the symmetry of the whole, such as the hypercube (a four-dimensional cube), whose symmetries are so numerous that it is hard to count them all. (But with proper guidance, young children using a simple pea-and-toothpick model can do it.) Other times it is the symmetry of the parts, as in the growth of natural objects from repetitive patterns of molecules or cells. In still other cases it is symmetry broken, as in the buckling of a cylindrical beam or the growth of a fertilized egg to a (slightly) asymmetrical adult animal. Unlike measurement, symmetry is seldom studied much in school at any level, yet it is equally fundamental as a model for explaining features of such diverse phenomena as the basic forces of nature, the structure of crystals, and the growth of organisms. Learning to recognize symmetry trains the mathematical eye.

VISUALIZATION recurs in many examples in this volume and is one of the most rapidly growing areas of mathematical and scientific research. The first step in data analysis is the visual display of data to search for hidden patterns. Graphs of various types provide visual display of relations and functions; they are widely used throughout science and industry to portray the behavior of one variable (e.g., sales) that is a function of another (e.g., advertising). For centuries artists and map makers have used geometric devices such as projection to represent three-dimensional scenes on a two-dimensional canvas or sheet of paper. Now computer graphics automate these processes and let us explore as well the projections of shapes in higher-dimensional space. Learning to visualize mathematical patterns enlists the gift of sight as an invaluable ally in mathematical education.

ALGORITHMS are recipes for computation that occur in every corner of mathematics. A common iterative procedure for projecting population growth reveals how simple orderly events can lead to a variety of behaviors—explosion, decay, repetition, chaos. Exploration of combinatorial patterns in geometric forms enables students to project geometric structures in higher dimensions where they cannot build real models. Even common elementary school algorithms for arithmetic take on a new dimension when viewed from the perspective of contemporary mathematics: rather than stressing the mastery of specific algorithms—which are now carried out principally by calculators or computers—school mathematics can instead emphasize more fundamental attributes of algorithms (e.g., speed, efficiency, sensitivity) that are essential for intelligent use of mathematics in the computer age. Learning to think algorithmically builds contemporary mathematical literacy.

Many other connective themes recur in this volume, including linkages of mathematics with science, classification as a tool for understanding, inference from axioms and data, and—most importantly—the role of exploration in the process of learning mathematics. Connections give mathematics power and help determine what is fundamental. Pedagogically, connections permit insight developed in one strand to infuse into others. Multiple strands linked by strong interconnections can develop mathematical power in students with a wide variety of enthusiasms and abilities.

GAINING PERSPECTIVE

Newton credited his extraordinary foresight in the development of calculus to the accumulated work of his predecessors: "If I have seen further it is by standing on the shoulders of giants." Those who develop mathematics curricula for the twenty-first century will need similar foresight.

Not since the time of Newton has mathematics changed as much as it has in recent years. Motivated in large part by the introduction of computers, the nature and practice of mathematics have been fundamentally transformed by new concepts, tools, applications, and methods. Like the telescope of Galileo's era that enabled the Newtonian revolution, today's computer challenges traditional views and forces re-examination of deeply held values. As it did three centuries ago in the transition from Euclidean proofs to Newtonian analysis, mathematics once again is undergoing a fundamental reorientation of procedural paradigms.

Examples of fundamental change abound in the research literature of mathematics and in practical applications of mathematical methods. Many are given in the essays in this volume:

- Uncertainty is not haphazard, since regularity eventually emerges.
- Deterministic phenomena often exhibit random behavior.
- Dimensionality is not just a property of space but also a means of ordering knowledge.
- Repetition can be the source of accuracy, symmetry, or chaos.
- Visual representation yields insights that often remain hidden from strictly analytic approaches.
- Diverse patterns of change exhibit significant underlying regularity.

By examining many different strands of mathematics, we gain perspective on common features and dominant ideas. Recurring concepts (e.g., number, function, algorithm) call attention to what one must know in order to *understand* mathematics; common actions (e.g., represent, discover, prove) reveal skills that one must develop in order to *do* mathematics. Together, concepts and actions are the nouns and verbs of the language of mathematics.

What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. To grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections.

The essays in this volume provide five extended case studies that exemplify how this can be done. Other authors could just as easily have described five or ten different examples. The books and articles listed below are replete with additional examples of rich mathematical ideas. What matters in the study of mathematics is not so much which particular strands one explores, but the presence in these strands of significant examples of sufficient variety and depth to reveal patterns. By encouraging students to explore patterns that have proven their power and significance, we offer them broad shoulders from which they will see farther than we can.

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